

# Incorporating Liquidity Risk in Value-at-Risk Based on Liquidity Adjusted Returns\*

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## Abstract

In this paper, based on Acharya and Pedersen's [*Journal of Financial Economics* (2006)] overlapping generation model, we show that liquidity risk could influence the market risk forecasting through at least two ways. Then we argue that traditional liquidity adjusted VaR measure, the simply adding of the two risk measure, would underestimate the risk. Hence another approach, by modeling the liquidity adjusted returns ( $LAr$ ) directly, was employed to incorporate liquidity risk in VaR measure in this study. Under such an approach, China's stock market is specifically studied. We estimate the one-day-ahead "standard" VaR and liquidity adjusted VaR by forming a skewed Student's t AR-GJR model to capture the asymmetric effect, non-normality and excess skewness of return, illiquidity and  $LAr$ . The empirical results support our theoretical arguments very well. We find that for the most illiquidity portfolio, liquidity risk represents more than 22% of total risk. We also find that simply adding of the two risk measure would underestimate the risk. The accuracy testing show that our approach is more accurate than the method of simply adding.

**JEL Classification Codes:** G11; G12; G18

**Keywords:** Value-at-Risk(VaR); Liquidity risk; Liquidity adjusted returns; Skewed Student's t; GJR model

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# 在VaR模型中嵌入流动性风险：基于流动性 调整收益率的方法\*

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## 摘 要

本文在Acharya and Pedersen (2006)的跨纪叠代模型基础上,证明了流动性风险可以通过至少两种方式影响市场风险。据此,我们进一步论证,通过将市场风险和流动性风险简单相加而获得流动性调整的VaR的方法会低估风险。因此,我们在这篇文章中直接对流动性调整收益率建模,从而预测流动性调整的VaR,并用这种方法研究了中国股票市场。为控制收益率序列、非流动性成本序列和流动性调整收益率序列中的非对称性、非正态性和偏性特点,我们建立了一个偏t分布的AR-GJR模型预测“标准”VaR和流动性调整的VaR。实证结果很好的支持了我们理论模型的预测。我们发现对流动性最差的资产组合而言,流动性风险可以占到总风险的22%。我们还发现简单相加的办法确实低估了风险。统计检验的结果证实我们的方法比简单相加的方法更为精确。

**JEL 分类号:** G11; G12; G18

**关键词:** 在险价值(VaR); 流动性风险; 流动性调整收益率; 偏t分布; GJR模型

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# 1 Introduction

Value-at-Risk (VaR) has been widely used in risk measurement by many financial institutions. But a fundamental assumption underlying the traditional VaR models is that assets can be traded in a liquid market. In reality, however, the capital market is not as liquid as we expect. Investors face not only the market risk but also the liquidity risk. Moreover, the bankruptcy of Long-Term Capital Management (LTCM) tell us that the illiquidity “is a big source of risk to an investor”<sup>1</sup>. Hence, it is necessary to incorporate liquidity risk in the VaR measure.

Few studies has focused on this issue<sup>2</sup>. For instance, Bangia *et al.* (1999) estimate the worst increase the bid-ask spread may suffer. They add it to the “standard” VaR and then obtain a liquidity adjusted VaR measure. More recently, Angelidis and Benos (2006) estimate a trade volume dependent model based on the components of the bid-ask spread and then incorporate a parametric liquidity risk in “standard” VaR measure. However, the most of these researches modeling the market risk and liquidity risk separately, and then add the two risk measure—value at market risk and value at liquidity risk. But recent development in asset pricing and market microstructure theories point out that liquidity risk should be priced (see, for example, Amihud and Mendelson, 1986; Holmström and Tirole, 2001; Acharya and Pedersen, 2006). Most empirical evidences also show that illiquidity securities should have higher expected returns (see, for instance, Brennan *et al.*, 1998; Amihud, 2002; Pástor and Stambaugh, 2003). Hence, it maybe inaccurate in calculating liquidity adjusted VaR if we omit the relation between liquidity risk and market risk.

In this paper, based on Acharya and Pedersen’s (2006) overlapping generation (OLG) model, we show that liquidity risk can influence the market risk forecasting through two ways. Then the accuracy of simply adding of the two risk measure would also be influenced through two ways: 1) First, as been shown in existed literatures, returns are low when illiquidity increases. Therefore, the value at market risk would increase and the simply adding would underestimate the risk. 2) Second, we show that the volatility of returns would be amplified if the volatility of illiquidity risk is large. Then the value at market risk would also increase and the simply adding would underestimate the risk, too. At a word, just add the two risk measure would underestimate the liquidity adjusted VaR.

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<sup>1</sup>The Economist, September 23, 1999.

<sup>2</sup>See the section 1 of Angelidis and Benos(2006) for a nice survey.

Hence, we employ another method to incorporate liquidity risk in VaR measure. We modeling the liquidity adjusted returns ( $LAr$ ) directly, where the  $LAr$  is equal to returns minus illiquidity cost. Since  $LAr$  itself considering the market risk and the liquidity risk simultaneously, this approach can avoid the underestimate problem in the simply adding of value at market risk and value at liquidity risk. Moreover,  $LAr$  satisfies Artzner *et al*'s (1999) "future value" viewpoint because it is the actually value of one unit of assets when investors need to liquidate the assets.

Under such an approach, China's stock market is specifically studied, a market considered as a very important emerging market. We form a skewed Student's  $t$  AR-GJR model to estimate the one-day-ahead "standard" VaR and liquidity adjusted VaR. We find that for the most illiquidity portfolio, liquidity risk represents more than 22% of total risk. We also find that simply adding of the two risk measure would underestimate the risk. The accuracy testing based on Kupiec's (1995) statistic show that our approach is more accurate than the method of simply adding.

Three main contributions are belong to this paper. Firstly, we propose a more accurate approach to modeling liquidity adjusted VaR. Secondly, this study adds to the evidence on the importance of liquidity risk in VaR measure. Lastly, there are rarely studies considering China's stock market about such issue in international journals. But China has became the most important emerging market and its stock market has been opened to international investors. Hence we need more researches, such as this paper, to study the characteristic of the risk in China's stock market.

The remainder of this paper is organized as follows. In Section 2, we will show the inaccuracy of the method of simply adding. Section 3 presents the data and econometric model. In Section 4, we will report the empirical results. Section 5 concludes the paper.

## 2 Theoretic framework

### 2.1 market risk and liquidity risk

In this subsection, we would find the relation between liquidity risk and market risk by simplifying Acharya and Pedersen's (2006) OLG model. The model assumes generation  $t$ , which is born at time  $t \in \{\dots, -2, -1, 0, -1, 2, \dots\}$  and lives in period  $t$  and  $t + 1$ , consists of  $N$  agents. Agent  $n$  of generation  $t$  only has an endowment  $e_t^n$  in period  $t$ . We assume he trades in period  $t$  and  $t + 1$  and maximizes his expect utility function  $-E_t \exp(-\gamma x_{t+1})$ , where  $\gamma$  is his constant absolute risk aversion and

$x_{t+1}$  is his consumption at time  $t + 1$ . Since there is no other source of income, the utility function is equal to  $-E_t \exp(-\gamma W_{t+1})$ , where  $W_{t+1}$  is the derived wealth by trading.

Suppose there are two kinds of asset, the risky asset with total of  $S$  shares and the riskfree asset. At time  $t$ , the risky asset has a share price of  $P_t$ , and has a per-share illiquidity cost of  $C_t$ . Hence, agents can buy the risky asset at  $P_t$  but must sell it at  $P_t - C_t$ .  $P_t$  and  $C_t$  are both random variables. Uncertainty about  $P_t$  is what generates the market risk in this model. Similarly, uncertainty about  $C_t$  generates the liquidity risk. Moreover, the illiquidity cost  $C_t$  is assumed to be autoregressive process of order one, that is

$$C_t = \bar{C} + \rho^C(C_{t-1} - \bar{C}) + \eta_t, \quad (1)$$

where  $\bar{C} \in \mathbb{R}_+$ ,  $\rho^C \in [0, 1]$ , and  $\eta_t$  is an independent identically distributed process with zero mean and variance  $Var(\eta_t) = \Sigma^C$ .

We assume the gross return of the riskfree asset is  $R^f$  ( $R^f > 1$ ). For the the risky asset's (net) return

$$r_t = \frac{P_t}{P_{t-1}} - 1 \quad (2)$$

and its relative illiquidity cost

$$c_t = \frac{C_t}{P_{t-1}}, \quad (3)$$

we can obtain two implications<sup>3</sup>:

**Proposition 1.** <sup>4</sup> *Returns are low when illiquidity increases,*

$$Cov_t(c_{t+1}, r_{t+1}) < 0. \quad (4)$$

There are lots of empirical evidences consistent with this proposition both in developed markets and emerging markets. For examples, Amihud (2002) finds a negative relation between the return on size portfolios traded in NYSE and their corresponding unexpected illiquidity; Bekaert *et al.*(2007) finds a negative relation between the returns and illiquidity for emerging markets.

**Proposition 2.** *The conditional variance of returns increases with the conditional variance of illiquidity,*

$$\frac{\partial Var_t(r_{t+1})}{\partial Var_t(C_{t+1})} > 0. \quad (5)$$

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<sup>3</sup>See appendix for proves.

<sup>4</sup>This proposition is similar to Proposition 3 in Acharya and Pedersen(2006).

The relation between the second moments of returns and illiquidity always be omitted in literatures. However, it is important to consider variance in risk measurement. Practically, we would find that both kinds of relation could influence the accuracy of the simply adding of the two risk measure.

## 2.2 Incorporating liquidity risk in VaR

Value-at-Risk (VaR) is defined as the worst outcome that is expected to occur over a predetermined period and at a given confidence level (say  $1 - \alpha$ ). The traditional VaR measure,  $\text{VaR}(r)$ , focuses on the market risk but doesn't consider the liquidity risk. In fact, we should compute

$$\text{VaR}(r + (-c)) \quad (6)$$

if we want to incorporate liquidity risk in VaR. Using  $-c$  here because  $c$  is defined as illiquidity in the assumption. In most of literatures, the market risk and the liquidity risk were modeled separately. The liquidity adjusted VaR was calculated by simply adding of the two risk measure, that is

$$\text{VaR}(r + (-c)) = \text{VaR}(r) + \text{VaR}(-c). \quad (7)$$

But as shown above, there exit at least two kinds of relation between the liquidity risk and the market risk. We now analyze the impact of the two kinds of relationship on the above method. Without loss generality, we focus on the one-step-ahead VaR computed in time  $t$

$$\text{VaR}_t(r_{t+1} + (-c_{t+1})) = \text{VaR}_t(r_{t+1}) + \text{VaR}_t(-c_{t+1}). \quad (8)$$

Firstly, according to proposition 1, we have  $\text{Cov}_t(c_{t+1}, r_{t+1}) < 0$ . Since  $\text{VaR}_t(-c_{t+1})$  refers to the worst increase the illiquidity may suffer, the probability that  $r_{t+1}$  would be low will increase when we consider the liquidity risk. Therefore, if we isolate the calculating of  $\text{VaR}_t(r_{t+1})$  from the calculating of  $\text{VaR}_t(-c_{t+1})$ , we would underestimate the risk. Contrarily, since  $\text{VaR}_t(r_{t+1})$  is defined as the worst outcome the return may occur, the probability that  $c_{t+1}$  would be high will increase when we consider the VaR measure of the market risk. Hence, we would also underestimate the risk if we isolate the calculating of  $\text{VaR}_t(-c_{t+1})$  from the calculating of  $\text{VaR}_t(r_{t+1})$ .

Secondly, according to proposition 2, we have  $\frac{\partial \text{VaR}_t(r_{t+1})}{\partial \text{VaR}_t(C_{t+1})} > 0$ . We know that the one-step-ahead VaR measure of the market risk is computed as

$$\text{VaR}_t(r_{t+1}) = \mu_{t+1}^r + z_\alpha^r \sigma_{t+1}^r, \quad (9)$$

where  $\mu_{t+1}^r$  is the conditional mean of the asset's return and  $\sigma_{t+1}^r$  the conditional standard variance of the return.  $z_\alpha^r$  is the left quantile at  $\alpha$  for the empirical distribution of the return. Since  $\sigma_{t+1}^r$  increases with the conditional standard variance of the illiquidity risk,  $\text{VaR}_t(r_{t+1})$  would be high (in absolute value) when we consider the liquidity risk<sup>5</sup>. Hence, we would underestimate the risk if we isolate the calculating of  $\text{VaR}_t(r_{t+1})$ .

To sum up, the simply adding of the two risk measure would underestimate the liquidity adjusted VaR. We suggest that it is more accurate to model the liquidity adjusted returns ( $LAr$ ) directly, where  $LAr$  is equal to returns minus illiquidity cost. Since  $LAr$  itself considering the market risk and the liquidity risk simultaneously, this approach can avoid the underestimate problem in the simply adding of value at market risk and value at liquidity risk. Moreover,  $LAr$  satisfies Artzner's (1999) "future value" viewpoint because it is the actually value of one unit of assets when investors need to liquidate the assets. Then we should calculate  $\text{VaR}(LAr)$  in this paper. To highlight the importance of the liquidity risk, we define and compute the relative liquidity risk proportion

$$\ell = \frac{\text{VaR}(LAr) - \text{VaR}(r)}{\text{VaR}(LAr)}. \quad (10)$$

### 3 Data and econometric models

China's stock market is specifically studied in this paper, transaction data cover the period from 2 January 2001 to 31 December 2008. Before describing our data set in detail, we first introduce the illiquidity measure used in this study.

#### 3.1 The illiquidity measure

The concept of (il)liquidity is elusive. Literatures about liquidity focus on one kind or several kinds of liquidity proxy because it is not observed directly. For examples, Amihud and Mendelson (1986) use the bid-ask spread relating to trading cost; Pástor and Stambaugh (2003) form a monthly liquidity measure by regressing individual stock's return minus the market return on the lagged individual stock's return and the lagged signed dollar trading volume using daily data; Amihud (2002) defined illiquidity as the average ratio of the daily absolute return to the dollar trading volume on that day; Bekaert *et al.*(2007) construct the proportion of zero

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<sup>5</sup>If we don't consider the liquidity risk, it is same to assume the mean and variance of illiquidity are both equal to zero.

daily returns observed over the relevant month for emerging market as liquidity measure. In the literatures of liquidity adjusted VaR, bid-ask spread is a widely used illiquidity measure. For instance, Bangia *et al.* (1999) classify illiquidity into the exogenous illiquidity and the endogeneous illiquidity and employ the bid-ask spread to represent the former. Based on the components of the bid-ask spread, Angelidis and Benos (2006) use order-based proxies of liquidity. In emerging markets, however, detailed transaction data of the bid-ask spreads are not widely available, especially for long time series. Hence, we employ Amihud (2002)'s illiquidity measure using only daily data. Particularly, the illiquidity of stock  $i$  in day  $t$  is

$$ILLIQ_t^i = \frac{|r_t^i|}{V_t^i}, \quad (11)$$

where  $r_t^i$  and  $V_t^i$  are the return and yuan<sup>6</sup> volume (in ten millions) for stock  $i$  on day  $t$ , respectively.

This illiquidity measure has been widely used in empirical studies, and has been shown to be a valid instrument for the illiquidity.  $ILLIQ$  is positively related to price impact to capture the price reaction to trading volume (Liu, 2006). Hasbrouck (2002) finds that the Spearman (Pearson) correlation between  $ILLIQ$  and a measure of Kyle's (1985) lambda is 0.737 (0.473) in the USA. Yuan volume in ten millions means that we assume the investor's position is ten millions yuan, and the illiquidity cost is positively related to trade demands. Therefore,  $ILLIQ$  captures both the exogenous illiquidity and the endogeneous illiquidity in Bangia *et al.* (1999).

$ILLIQ$ , in terms of return impact, can be viewed as the cost of 10 millions yuan trade. But China's stock market experiences a rapidly growth in the sample period—the market capitalizations of the market portfolio increase by almost 832 percent from January 2001 to December 2008<sup>7</sup>. Obviously, 10 millions yuan trade was more substantial in January 2001 than December 2008, so  $ILLIQ$  tend to be smaller in magnitude later in the period. Hence follow Pástor and Stambaugh (2003), we construct the scaled series  $(m_h/m_1)ILLIQ_t^i$ , where  $m_h$  is the total value of the market at the end of month  $h$  corresponding to day  $t$ , and  $m_1$  is the total value of the market at the end of January 2001. Finally, the illiquidity measure for stock  $i$  at day  $t$  we used is

$$c_t^i = \min\left\{\frac{m_h}{m_1}ILLIQ_t^i, 10.00\right\}. \quad (12)$$

<sup>6</sup>Yuan is the units of Renminbi (RMB, the Chinese currency).

<sup>7</sup>The total value of the market at the end of January 2001 and at the end of December 2008 are 1567768 and 14602379 millions yuan, respectively.



### 3.2 Data

China has two stock exchanges, the Shanghai stock exchange (SHSE) and the Shenzhen stock exchange (SZSE). The two were both inaugurated in the early 1990s, and the SZSE is relatively smaller. Chinese company can raise funds through an A or B share listing on one of the two exchanges. The A shares are held by Chinese citizens and purchased in RMB, while B shares are held by foreign parties and denominated in U.S. dollars. Since only a few firms offer B shares and the B shares always experience a light trading, we focus on the A shares only. In the remainder of this paper, the A shares market of the SHSE and the A shares market of the SZSE are, respectively, abbreviated to SHSE-A and SZSE-A.

The data used in this paper for the two stock exchanges are from the CSMAR China Stock Market Trading Database, which imitates CRSP and be widely used by Chinese academe and financial companies. For each year  $t$  (2001-2008), we allocate all the firms listed in the SHSE-A and the SZSE-A into five size-portfolios (from small to big: S1, S2, S3, S4 and S5) based on their market capitalization at the end of December of year  $t - 1$ . For all stocks, the days with no trading have been eliminated from the sample. Value-weight daily returns (in percent) and illiquidity costs (in percent) on the portfolios are calculated from the first trading day to the last trading day of year  $t$ . Particularly, for each portfolio  $p$  ( $p \in \{S1, S2, S3, S4, S5\}$ ), its return at day  $t$  is

$$r_t^p = \sum_{i \text{ in } p} w_t^i r_t^i, \quad (13)$$

and the illiquidity cost at day  $t$  is

$$c_t^p = \sum_{i \text{ in } p} w_t^i c_t^i, \quad (14)$$

where  $w_t^i$  are value-based weights. Suppose month  $h$  is corresponding to the day  $t$ , then we form the weights based on the market value for firm  $i$  at the end of month  $h - 1$ . Similarly, we compute the  $LAr$  (in percent) of portfolio  $p$  at day  $t$ , as

$$\begin{aligned} LAr_t^p &= \sum_{i \text{ in } p} w_t^i LAr_t^i \\ &= \sum_{i \text{ in } p} w_t^i (r_t^i - c_t^i). \end{aligned} \quad (15)$$

The number of valid observation days in the sample for each portfolio is 1932. We plot  $r_t^p$ ,  $c_t^p$  and  $LAr_t^p$  in figure 1, figure 2 and figure 3, respectively. These

figures indicate that both returns time series and illiquidity costs time series exhibit volatility clustering. So, not surprisingly, the  $LAr$  time series, which is the differences between the returns and the illiquidity costs, also exhibit volatility clustering. The high volatility in magnitude later may due to the reform of non-tradable shares and the subprime crisis. Moreover, the five portfolios experience similar price trend, that is periods of low volatility and periods of high volatility are almost the same in the five portfolios. This finding is consistent with Morck *et al.*'s (2000) conclusion. They find that the systematic component of returns variation is large in emerging markets, including China.

[Figure 1 about here.]

[Figure 2 about here.]

[Figure 3 about here.]

Table 1 presents descriptive statistics for the five size-portfolios. The portfolios have almost the same average number of firms. Obviously, portfolio contain smaller firms has higher illiquidity cost, and S5 exhibits superior liquidity level. But the mean of the returns is nearly indifference among the five portfolios:  $E(r)$  for S5 is just a little higher than for S1. For each portfolio, the covariance between the returns and illiquidity costs are negative, supporting the proposition 1 we have shown. We also find that this covariance is larger (in absolute value) for less liquid portfolios. This finding indicates that liquidity risk is more influential for less liquid assets, which is familiar in reality. The size-portfolio with smaller firms has both higher return volatility and higher illiquidity volatility than portfolio contain bigger firms except for S1. This is the evidence to prove proposition 2.

[Table 1 about here.]

The liquidity adjusted returns for the high capitalization stocks (S5) outperformance the low capitalization stocks (S1) clearly. Therefore, liquidity risk is an important non-market risk we should consider. More specifically, the returns for other 4 size-portfolios exhibit negatively skewed except for S5. In contrast, illiquidity for all the portfolios exhibit positive skewed. After adjusting for illiquidity,  $LAr$  for all the five portfolios show positive skewed.

### 3.3 Econometric models

VaR is an estimation of the tails of the empirical distribution (Angelidis *et al.*, 2004). The family of ARCH models are popularly used in modeling the daily VaR, such as RiskMetrics<sup>TM</sup> or GARCH(1, 1) under specific distribution. We would also use ARCH models in this study. Particularly, we set the conditional mean equation by fitting an  $p$  orders autoregressive (AR( $p$ )) process, that is

$$y_t = \phi_0 + \sum_{i=1}^p \phi_i y_{t-i} + \varepsilon_t, \quad (16)$$

where  $y_t$  is the return series, illiquidity series or *LAr* series.

For many stocks, the “bad” news and the “good” news have different pronounced effect on volatility— the so-called asymmetric effects. This asymmetric effects is important in the accuracy of VaR estimation. Brooks and Persaud (2003) find that the VaR would be underestimated if the models leave asymmetric effects out of account. Hence, we employ the GJR model, which was introduced by Glosten, Jagannathan and Runkle (1993), to capture the asymmetric effects. On the other hand, the Jarque-Bera tests in Tabel 1 show that all the series exhibit a non-normal distribution. And all the series show either positively skewed or negatively skewed. Therefore, to account for the non-normality and the excess skewness, we, follow Giot and Laurent (2003), assume the residuals  $\varepsilon_t$  of the conditional mean equation (16) have a skewed Student’s t-distribution. Finally, we set the conditional variance equation to be a skewed Student’s t GJR model, that is

$$\varepsilon_t = \sigma_t z_t \quad (17)$$

$$\sigma_t^2 = \omega + \alpha_1 \varepsilon_{t-1}^2 + \gamma_1 \lambda_{t-1} \varepsilon_{t-1}^2 + \beta_1 \sigma_{t-1}^2, \quad (18)$$

where  $\lambda_t$  is a dummy variable that take the value 1 when  $\varepsilon_t$  is negative and 0 when it is positive. If  $\gamma_1 > 0$ , negative shocks will have larger effects on volatility than positive shocks— the so-called leverage effects.

Follow Lambert and Laurent (2001) and Giot and Laurent (2003), the innovation process  $z_t$  is assumed to be (standardized) skewed Student distributed, that is

$$f(z|\xi, \nu) = \begin{cases} \frac{2}{\xi + \frac{1}{\xi}} sg[\xi(sz + m)|\nu], & \text{if } z < -\frac{m}{s}, \\ \frac{2}{\xi + \frac{1}{\xi}} sg[(sz + m)/\xi|\nu], & \text{if } z \geq -\frac{m}{s}, \end{cases} \quad (19)$$

$$(19')$$

where  $g(\cdot|\nu)$  is the standard Student's t density with freedom  $\nu$  and  $\xi$  is the asymmetry coefficient: the density is skew to the right (left), if  $\log(\xi) > 0 (< 0)$ <sup>8</sup>.  $m$  and  $s^2$  are respectively the mean and the variance of the non-standardized skewed Student's t distribution:

$$m = \frac{\Gamma\left(\frac{\nu-1}{2}\right)\sqrt{\nu-2}}{\sqrt{\pi}\Gamma\left(\frac{\nu}{2}\right)}\left(\xi - \frac{1}{\xi}\right), \quad (20)$$

$$s^2 = \left(\xi^2 + \frac{1}{\xi^2} - 1\right) - m^2. \quad (21)$$

To estimate VaR, we need know the quantile function of the distribution. Lambert and Laurent (2000) show that the quantile function of the standardized skewed Student's t distribution is

$$skst_{\alpha,\nu,\xi} = \frac{skst_{\alpha,\nu,\xi}^* - m}{s}, \quad (22)$$

in which,

$$skst_{\alpha,\nu,\xi}^* = \begin{cases} \frac{1}{\xi}st_{\alpha,\nu}\left[\frac{\alpha}{2}(1+\xi^2)\right], & \text{if } \alpha < \frac{1}{1+\xi^2}, \\ -\xi st_{\alpha,\nu}\left[\frac{1-\alpha}{2}(1+\xi^{-2})\right], & \text{if } \alpha \geq \frac{1}{1+\xi^2}, \end{cases} \quad (23)$$

$$(23')$$

where  $st_{\alpha,\nu}$  is the quantile function of the standard Student's t density. Then given confidence level  $1 - \alpha$ , the one-day-ahead VaR estimation is given by

$$VaR_t(y_{t+1}) = E_t(y_{t+1}) + skst_{\alpha,\nu,\xi}\sigma_{t+1}. \quad (24)$$

## 4 Empirical results

Based on the estimation of the parametric model in subsection 3.3, we calculate and compare the two kinds liquidity adjusted VaR—  $VaR(LAr)$ , and the simply adding of  $VaR(r)$  and  $VaR(-c)$ . In addition, we will compute the liquidity component in  $VaR(LAr)$  to highlight the importance of liquidity risk.

### 4.1 Skewed Student's t AR-GJR model estimation

In this subsection, we firstly estimate the skewed Student's t AR-GJR model (16)-(21) for the five size-portfolios' returns, illiquidity costs and  $LAr$ , respectively. All the econometric models in this paper was estimated by G@RCH 5.0, an Ox package. We only report the results for the volatility specification. Table 2 provides the

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<sup>8</sup>See Giot and Laurent (2003) for a more detailed discussion.

parameters' estimation for returns and illiquidity costs. we also compute the Pearson correlation coefficient between the conditional variances of return and illiquidity.

[Table 2 about here.]

The returns of five size-portfolios feature relatively similar volatility characteristics. The conditional variances exhibit strong memory effects since the autoregressive coefficient  $\beta_1$  is nearly 0.9.  $\gamma_1$  for returns is positive and significant for all portfolios, which implies that the leverage effect for negative returns also exists in China's stock market.  $\log(\xi)$  is negative for all portfolios and significant except for S5, indicating a negative excess skewness.

For the illiquidity costs, memory effects of the conditional variance also exist in each portfolio. But S5 show a high sensitivity to short run shock ( $\alpha_1 = 0.883$ ).  $\log(\xi)$  is positive and significant for all portfolios, indicating a positive excess skewness.  $\gamma_1$  is negative and significant for each portfolio. Notice that a negative illiquidity shock equal to a positive liquidity shock. Hence, there also exist a leverage effect for negative liquidity shocks. Moreover, the absolute value of  $\gamma_1$  for illiquidity costs is much larger than for returns, which implies that liquidity is much more sensitive. The bankruptcy of LTCM is a suitable example. "...LTCM's partners, calling in from Tokyo and London, reported that their markets has dried up. There were no buyers, no sellers. ..."<sup>9</sup>.

The Pearson correlation coefficient between the conditional variances of returns and illiquidity costs is positive for all portfolios, supporting the proposition 2 we have proved. Moreover, this correlation coefficient is larger for less liquid portfolios, indicating that liquidity risk is more influential for less liquid assets, similar to the conclusion in table 1.

Table 3 presents the estimation results for the conditional variance models of LAr. All the portfolios exhibit relatively similar volatility characteristics for LAr. The autoregressive coefficient  $\beta_1$  is close to 0.9, pointing out a memory effect of the conditional variance for LAr.  $\gamma_1$  for LAr is positive and significant for each portfolio, indicating a leverage effect for negative LAr in the conditional variance specification.  $\log(\xi)$  is negative and significant for all portfolios, which implies a negative excess skewness.

[Table 3 about here.]

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<sup>9</sup>Wall Street Journal, November 16, 1998.

## 4.2 Liquidity adjusted VaR

In this subsection, we calculate and compare the two kinds one-day-ahead liquidity adjusted VaR—  $\text{VaR}(LAr)$ , and the simply adding of  $\text{VaR}(r)$  and  $\text{VaR}(-c)$ . We will also compute the liquidity component  $\ell$  in equation (10) to emphasize the importance of liquidity risk.

Table 4 provides that two kinds one-day-ahead liquidity adjusted VaR and the liquidity component  $\ell$  given  $\alpha = 5\%$  or  $1\%$  ( the confidence level  $1 - \alpha$  is 95% or 99%). We find that, without adjusting for liquidity risk, the  $\text{VaR}(r)$  is relatively similar for S1, S2, S3 and S4. S5 has a smaller  $\text{VaR}(r)$ . But after incorporating liquidity risk in VaR,  $\text{VaR}(LAr)$  is larger for more illiquidity (lower capitalized) portfolio. Also, S5 is much outperformance S1: the difference between  $\text{VaR}(LAr)$  of S1 and S5 is 1.583 percent when  $\alpha = 5\%$  or 2.371 percent when  $\alpha = 1\%$ , but it is 0.523 or 0.675 for  $\text{VaR}(r)$ . Moreover, the liquidity component  $\ell$  is more than 22% for the low capitalization portfolios and almost 5% even for the high capitalization ones. All the above results indicate that liquidity risk is even more important and we must incorporate the liquidity risk in VaR measure.

[Table 4 about here.]

By comparing  $\text{VaR}(LAr)$  with  $\text{VaR}(r)+\text{VaR}(-c)$ , we find that  $\text{VaR}(LAr)$  is a little larger than the simply adding for all portfolios whenever  $\alpha = 5\%$  or  $1\%$ . It appears that the simply adding  $\text{VaR}(r)+\text{VaR}(-c)$  underestimates the risk. However, we couldn't make the final judgement before testing the accuracy of the two approaches. We will do the test in next subsection.

## 4.3 Accuracy testing

The accuracy testing is based on the statistic developed by Kupiec (1995). For a  $T$ -day period, suppose  $N$  is the number of empirical failure days, which is the number returns exceed (in absolute value) the forecasted VaR. If the VaR model is correctly specified,  $N/T$  should be equal to the theoretical specified VaR level  $\alpha$ . Then the appropriate likelihood ratio statistic, under the null hypothesis that  $N/T = \alpha$ , is:

$$LR_{uc} = -2 \left[ \log \left( (1 - \alpha)^{T-N} \alpha^N \right) - \log \left( \left(1 - \frac{N}{T}\right)^{T-N} \left(\frac{N}{T}\right)^N \right) \right] \sim \chi_1^2 \quad (25)$$

Table 5 presents the number of failure days for the two kinds liquidity adjusted VaR model —  $\text{VaR}(LAr)$  and  $\text{VaR}(r)+\text{VaR}(-c)$  — for each portfolio in the sample

period. Based on equation (25), the 95% confidence intervals of the number of failure days are (77.82, 115.38) and (10.75, 27.89) for  $\alpha = 5\%$  and  $\alpha = 1\%$ , respectively. In all portfolios, whenever  $\alpha = 5\%$  or  $1\%$ , the number of failure days of the  $\text{VaR}(LAr)$  model is closer to the theoretical number than the model of simply adding. Generally, at the 95% confidence level,  $\text{VaR}(LAr)$  are more accurate than  $\text{VaR}(r)+\text{VaR}(-c)$ , even though the method of simply adding also generates adequate risk predictions.

[Table 5 about here.]

In addition, the number of failure days predicted by  $\text{VaR}(r)+\text{VaR}(-c)$  are closer to the expected number for S5 both at  $\alpha = 5\%$  and  $\alpha = 1\%$ . This finding supports our arguments that the relationships between liquidity risk and market risk produce the inaccuracy of the method of simply adding. In fact, we find that the negative covariance and the positive correlation between returns and illiquidity costs, which are the causations of the inaccuracy of the  $\text{VaR}(r)+\text{VaR}(-c)$  model, both are the smallest (in absolute value) for S5 in table 1 and table 2. Then it is not surprising that  $\text{VaR}(r)+\text{VaR}(-c)$  could predict more precise number of failure days for S5 than for other portfolios.

## 5 Conclusions

In this paper, by simplifying Acharya and Pedersen's (2006) overlapping generation model, we show that liquidity risk could influence the market risk forecasting through two ways. Then the accuracy of the traditional liquidity adjusted VaR measure, the simply adding of the two risk measure, would also be influenced through two ways: 1) First, returns are low when illiquidity increases. Therefore, the value at market risk would increase and the simply adding would underestimate the risk. 2) Second, we show that the volatility of returns would be amplified if the volatility of illiquidity risk is large. Then the value at market risk would also increase and the simply adding would underestimate the risk, too. At a word, just add the two risk measure would underestimate the liquidity adjusted VaR.

Hence, we employ another method to incorporate liquidity risk in VaR measure. We modeling the liquidity adjusted returns ( $LAr$ ) directly, where the  $LAr$  is equal to returns minus illiquidity cost. Under such an approach, China's stock market is specifically studied. We first construct a skewed Student's t AR-GJR model to capture the asymmetric effect, non-normality and excess skewness of return, illiquidity

and  $LAr$ . Then we estimate the one-day-ahead “standard” VaR and liquidity adjusted VaR. We find that for the most illiquidity portfolio, liquidity risk represents more than 22% of total risk. We also find that simply adding of the two risk measure would underestimate the risk. The accuracy testing based on Kupiec’s (1995) statistic show that our approach is more accurate than the method of simply adding.

The findings of this paper make three main contributions to literatures. Firstly, we propose a more accurate approach to modeling liquidity adjusted VaR. Secondly, this study adds to the evidence on the importance of liquidity risk in VaR measure. Lastly, there are rarely studies considering China’s stock market about such issue in international journals. But China has become the most important emerging market and its stock market has been opened to international investors. Hence we need more researches, such as this paper, to study the characteristic of the risk in China’s stock market.

## Appendix

We first solve the problem of investor  $n$  at time  $t$ . We assume the investor  $n$  purchases  $y^n$  shares of the risky asset. Then the agent’s problem is

$$\max_{y^n \in \mathbb{R}^+} E_t(W_{t+1}^n) - \frac{1}{2}\gamma Var_t(W_{t+1}^n), \quad (\text{A.1})$$

where

$$W_{t+1}^n = (P_{t+1} - C_{t+1})y^n + R^f(e_t^n - P_t y^n). \quad (\text{A.2})$$

From the first order condition, we have

$$y^n = \frac{1}{\gamma} [Var_t(P_{t+1} - C_{t+1})]^{-1} [E_t(P_{t+1} - C_{t+1}) - R^f P_t]. \quad (\text{A.3})$$

Since the total supply of risky asset  $S = \sum_n y^n$ , then we have equilibrium conditon

$$P_t = \frac{1}{R^f} [E_t(P_{t+1} - C_{t+1}) - \frac{\gamma S}{N} Var_t(P_{t+1} - C_{t+1})]. \quad (\text{A.4})$$

We can obtain the unique stationary linear equilibrium,

$$P_t = \Upsilon + \Phi C_t, \quad (\text{A.5})$$



where

$$\Upsilon = -\frac{1}{R^f - 1} \left[ \frac{R^f(1 - \rho^C)}{R^f - \rho^C} \bar{C} + \frac{\gamma S}{N} \text{Var}_t \left( -\frac{R^f}{R^f - \rho^C} \eta_t \right) \right], \quad (\text{A.6})$$

$$\Phi = -\frac{\rho^C}{R^f - \rho^C}. \quad (\text{A.7})$$

*Proof of Proposition 1*

The conditional covariance between illiquidity and the gross return is

$$\begin{aligned} \text{Cov}_t(c_{t+1}, R_{t+1}) &= \frac{1}{P_t^2} \text{Cov}_t(C_{t+1}, P_{t+1}) \\ &= \frac{1}{P_t^2} \text{Cov}_t \left( C_{t+1}, -\frac{\rho^C}{R^f - \rho^C} C_{t+1} \right) \\ &= \frac{1}{P_t^2} \left( -\frac{\rho^C}{R^f - \rho^C} \right) \text{Var}_t(C_{t+1}) \\ &< 0, \end{aligned} \quad (\text{A.8})$$

which yields the proposition.  $\square$

*Proof of Proposition 2*

The conditional variance of the gross return is

$$\begin{aligned} \text{Var}_t(R_{t+1}) &= \frac{1}{P_t^2} \text{Var}_t(P_{t+1}) \\ &= \frac{1}{P_t^2} \text{Var}_t \left( -\frac{\rho^C}{R^f - \rho^C} C_{t+1} \right) \\ &= \frac{1}{P_t^2} \left( \frac{\rho^C}{R^f - \rho^C} \right)^2 \text{Var}_t(C_{t+1}). \end{aligned} \quad (\text{A.9})$$

So we have

$$\frac{\partial \text{Var}_t(R_{t+1})}{\partial \text{Var}_t(C_{t+1})} = \frac{1}{P_t^2} \left( \frac{\rho^C}{R^f - \rho^C} \right)^2 > 0, \quad (\text{A.10})$$

which yields the proposition.  $\square$

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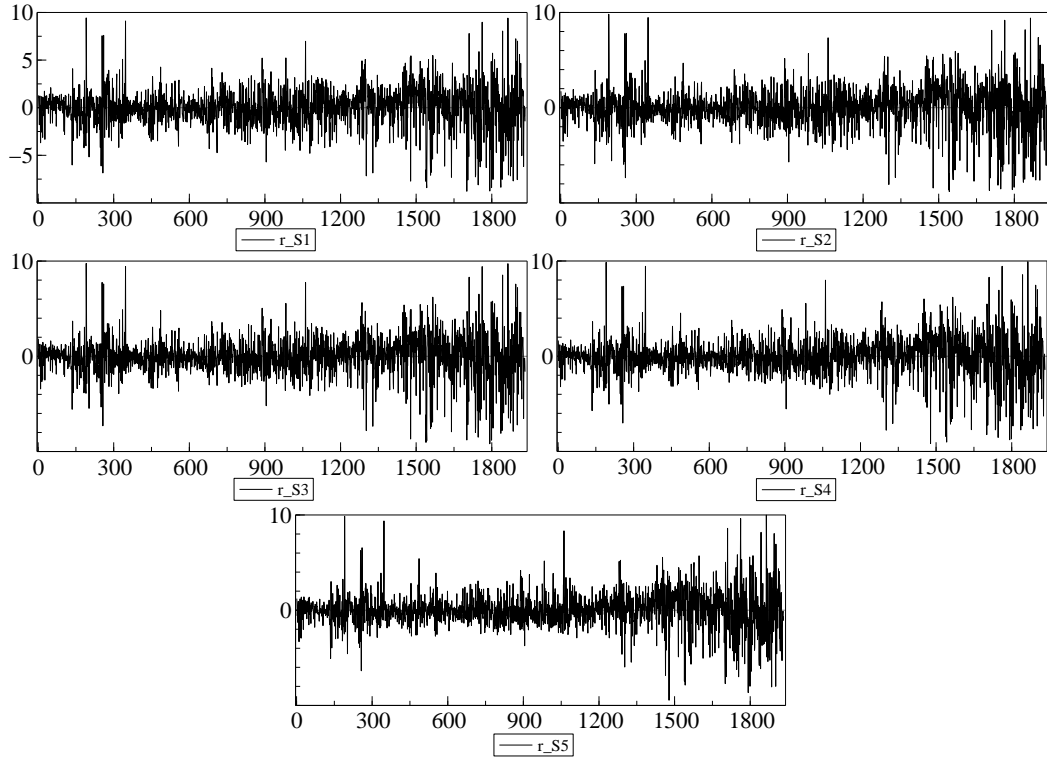


Figure 1. Daily Returns (in percent) of size-portfolios (from small to big: S1, S2, S3, S4 and S5) from 2 January 2001 to 31 December 2008.

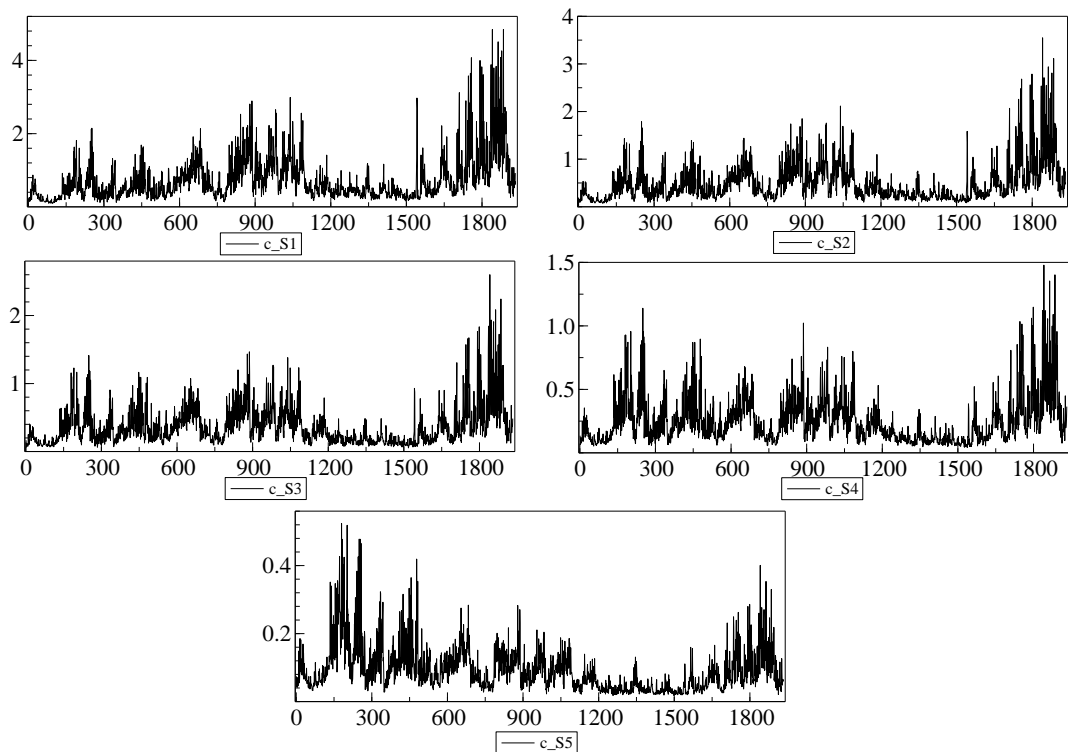


Figure 2. Daily illiquidity costs (in percent) of size-portfolios (from small to big: S1, S2, S3, S4 and S5) from 2 January 2001 to 31 December 2008.

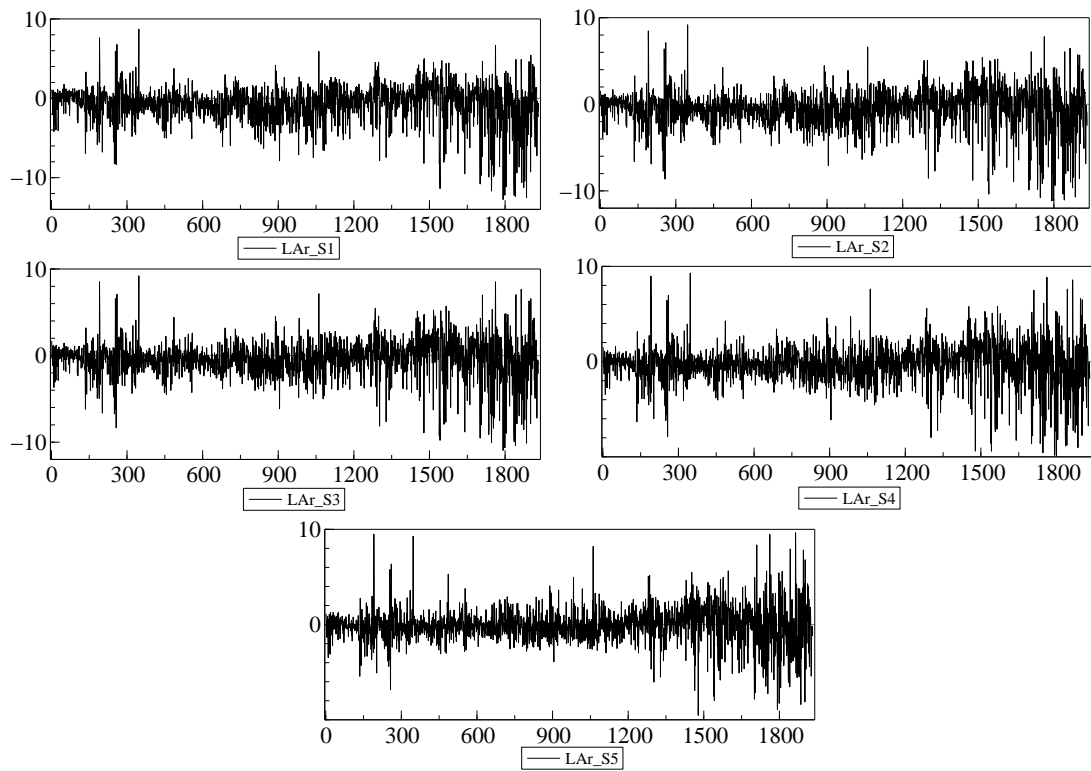


Figure 3. Daily Liquidity adjusted returns ( $LAr$ , in percent) of size-portfolios (from small to big: S1, S2, S3, S4 and S5) from 2 January 2001 to 31 December 2008.

Table 1: This table presents descriptive statistics for the five size-portfolios. We report the average number of firms for each portfolio in Panel A.  $E(\cdot)$ ,  $Std(\cdot)$  and  $Skew(\cdot)$  are, respectively, the mean, the standard variance and the Skewness for the time series. Jarque-Bera is the Jarque-Bera test statistics for normality.  $Cov(r, c)$  is the covariance between return and illiquidity. The sample period is from 2 January 2001 to 31 December 2008, including 1932 observations.

	S1	S2	S3	S4	S5
<i>Panel A. The average number of firms</i>					
	233.61	242.27	244.50	244.32	240.49
<i>Panel B. Descriptive statistics for returns and illiquidity costs</i>					
$E(r)$	0.0222	0.0209	0.0224	0.0208	0.0204
$Std.(r)$	2.1152	2.1446	2.1242	2.0729	1.8800
$Skew.(r)$	-0.3977	-0.3889	-0.3835	-0.3139	0.0228
Jarque-Bera	657.08	774.06	988.22	1062.3	1425.1
P-value	0.0000	0.0000	0.0000	0.0000	0.0000
$E(c)$	0.7345	0.4958	0.3662	0.2502	0.0931
$Std.(c)$	0.6056	0.4015	0.2851	0.1850	0.0679
$Skew.(c)$	2.4157	2.3300	2.1649	1.9721	2.0828
Jarque-Bera	7270.1	7024.0	5710.5	3671.3	4659.7
P-value	0.0000	0.0000	0.0000	0.0000	0.0000
$Cov(r, c)$	-0.4445	-0.2995	-0.2036	-0.1245	-0.0314
<i>Panel C. Descriptive statistics for liquidity adjusted returns</i>					
$E(LAr)$	-0.7122	-0.4749	-0.3438	-0.2293	-0.0727
$Std.(LAr)$	2.3937	2.3151	2.2362	2.1401	1.8979
$Skew.(LAr)$	-1.1971	-0.9151	-0.7628	-0.5689	-0.0568
Jarque-Bera	1545.4	1104.6	1095.7	1014.7	1284.8
P-value	0.0000	0.0000	0.0000	0.0000	0.0000

Table 2: This table presents the estimation results for the conditional variance models of returns and illiquidity costs. Robust standard errors are reported in parentheses.  $Corr(\sigma_r^2, \sigma_c^2)$  is the Pearson correlation coefficient between the conditional variances of returns and illiquidity. The sample period is from 2 January 2001 to 31 December 2008, including 1932 observations.

$y$	S1		S2		S3		S4		S5	
	$r$	$c$	$r$	$c$	$r$	$c$	$r$	$c$	$r$	$c$
$\omega$	0.062(0.024)	0.002(0.001)	0.051(0.021)	0.001(0.000)	0.042(0.017)	0.001(0.000)	0.037(0.016)	0.000(0.000)	0.031(0.013)	0.001(0.051)
$\alpha_1$	0.093(0.023)	0.242(0.030)	0.092(0.022)	0.235(0.027)	0.089(0.021)	0.228(0.025)	0.085(0.021)	0.218(0.025)	0.069(0.017)	0.443(0.067)
$\beta_1$	0.878(0.023)	0.856(0.016)	0.883(0.020)	0.868(0.014)	0.887(0.018)	0.869(0.013)	0.889(0.018)	0.876(0.014)	0.899(0.017)	0.777(0.024)
$\gamma_1$	0.045(0.022)	-0.324(0.040)	0.050(0.022)	-0.358(0.038)	0.052(0.022)	-0.342(0.036)	0.060(0.023)	-0.335(0.032)	0.070(0.023)	-0.441(0.066)
$\log(\xi)$	-0.174(0.029)	0.602(0.054)	-0.162(0.028)	0.610(0.048)	-0.142(0.029)	0.615(0.041)	-0.128(0.028)	0.629(0.038)	-0.032(0.029)	0.548(0.035)
$\nu$	6.827(1.095)	4.251(0.411)	5.212(0.654)	4.240(0.403)	5.769(0.804)	4.167(0.395)	5.560(0.748)	4.133(0.381)	5.212(0.654)	2.527(0.025)
$Corr(\sigma_r^2, \sigma_c^2)$	0.660		0.576		0.543		0.536		0.256	
Log-likelihood	-3883.364	37.526	-3882.678	688.234	-3831.761	1265.413	-3773.060	2069.560	-3575.176	3787.696



Table 3: This table presents the estimation results for the conditional variance model of LAr. Robust standard errors are reported in parentheses. The sample period is from 2 January 2001 to 31 December 2008, including 1932 observations.

	S1	S2	S3	S4	S5
$\omega$	0.117(0.038)	0.079(0.027)	0.059(0.021)	0.046(0.018)	0.033(0.014)
$\alpha_1$	0.092(0.030)	0.091(0.027)	0.088(0.024)	0.083(0.023)	0.067(0.017)
$\beta_1$	0.870(0.024)	0.880(0.021)	0.885(0.019)	0.887(0.018)	0.897(0.017)
$\gamma_1$	0.039(0.033)	0.043(0.029)	0.047(0.026)	0.060(0.026)	0.074(0.025)
$\log(\xi)$	-0.364(0.030)	-0.299(0.028)	-0.252(0.029)	-0.203(0.029)	-0.056(0.029)
$\nu$	5.693(0.777)	5.742(0.825)	5.827(0.846)	5.678(0.794)	5.298(0.687)
Log-likelihood	-3995.937	-3968.288	-3901.662	-3819.144	-3594.588

Table 4: This table presents the one-day-ahead liquidity adjusted VaR and the liquidity component  $\ell$ .  $\text{VaR}(r)$  is the one-day-ahead VaR without considering liquidity risk;  $\text{VaR}(LAr)$  is the liquidity adjusted VaR based on liquidity adjusted returns; the simply adding of  $\text{VaR}(r)$  and  $\text{VaR}(-c)$  is denoted as  $\text{VaR}(r)+\text{VaR}(-c)$ .

		S1	S2	S3	S4	S5
$\alpha = 5\%$	$\text{VaR}(r)$	4.410	4.347	4.500	4.382	3.887
	$\text{VaR}(LAr)$	5.713	5.363	5.325	4.958	4.130
	$\ell$	22.8%	18.9%	15.5%	11.6%	5.88%
	$\text{VaR}(r)+\text{VaR}(-c)$	5.659	5.343	5.215	4.888	4.022
$\alpha = 1\%$	$\text{VaR}(r)$	7.118	7.152	7.452	7.280	6.443
	$\text{VaR}(LAr)$	9.151	8.665	8.652	8.123	6.780
	$\ell$	22.2%	17.5%	13.9%	10.4%	4.97%
	$\text{VaR}(r)+\text{VaR}(-c)$	8.863	8.557	8.466	7.997	6.667

Table 5: This table presents the number of failure days for the  $\text{VaR}(LAr)$  model and the  $\text{VaR}(r)+\text{VaR}(-c)$  model. The sample period is from 2 January 2001 to 31 December 2008, including 1932 observations.

		S1	S2	S3	S4	S5
$\alpha = 5\%$	Theoretical Number	96.60	96.60	96.60	96.60	96.60
	$\text{VaR}(LAr)$	97	97	97	97	97
	$\text{VaR}(r)+\text{VaR}(-c)$	94	98	94	98	95
$\alpha = 1\%$	Theoretical Number	19.32	19.32	19.32	19.32	19.32
	$\text{VaR}(LAr)$	20	20	20	20	20
	$\text{VaR}(r)+\text{VaR}(-c)$	17	16	18	18	20