GARCH Option Pricing Models, the CBOE VIX and Variance Risk Premium

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Abstract

In this paper, we derive the corresponding implied VIX formulas under the locally risk-neutral valuation relationship proposed by Duan (1995) when various forms of GARCH model are proposed for S&P 500 index. The empirical study shows that the GARCH implied VIX is consistently and significantly lower than the CBOE VIX for all kinds of GARCH model investigated. Moreover, the magnitude of the difference suggests that the GARCH option pricing model is not capable of capturing the variance premium, which indicates the incompleteness of the GARCH option pricing under the locally risk-neutral valuation relationship. The source of this kind of incompleteness is then theoretically analyzed. It is shown that the framework of GARCH option pricing model fails to incorporate the price of volatility risk or variance premium.

1 Introduction

Finance literature has put much effort on studying the premia that the investors require for compensating various risks in financial market, especially the equity risk premium for price risk(volatility). However, instead of a constant volatility assumed in the Black-Scholes framework, a lot of research has confirmed that the volatility itself is time-varying, which is termed volatility risk. Many stochastic volatility models and various GARCH models go along this line.

Then what the finance literature concerns is whether this volatility risk is priced and compensated in financial market. One of the possible rationales for the existence of volatility risk premium is the negative correlation between the volatility and the index, which has been verified in many literature. In the context of asset pricing theory, the source of risk is the correlation with the market portfolio, aggregate consumption or pricing kernel. Theoretically, the negative correlation between volatility and index suggests a negative risk premium. If so, the premia required by investors should be reflected on the prices of volatility-dependent assets such as options and volatility products. The pursue of empirical evidence generally proceeds in two directions. One is to study the phenomenon that the implied volatility of options is higher than the realized volatility. Various delta-neutral portfolios of option are constructed to test whether significant gains or losses would be produced. Another one is to investigate the difference between the variance swap rate and the realized variance, which is coined variance premium. Variance swap rate, the risk-neutral expectation of the future variance, can be replicated with European options(See Demeterfi, Derman, Kamal and Zou, 1999; Carr and Wu, 2009). Methodologies for the calculating of a model-free realized variance have also been developed(see Andersen, 2008).

Since 1980s, the option pricing models with stochastic volatility had introduced the market price of volatility risk when changing from physical probability measure to risk-neutral measure. These papers include Wiggins (1987), Johnson and Shanno (1987), Hull and White (1987), Scott (1987) and Heston (1993). However, they set the market price of volatility risk either to zero or a constant and discussed little about the size, sign or dynamics of this parameter.

Since the beginning of this century, the evidence of the existence of volatility risk premium have been well documented. Coval and Shumway (2001) studied the expected option return under the framework of the classic asset pricing theory. They shown that the zero-beta, at the money straddles which are in long positions of volatility suffer from average weekly losses of about three percent. Bakshi and Kapadia (2003) constructed a correspondence between the sign and magnitude of the volatility risk premium and the mean delta-hedged portfolio returns. Their empirical results indicated a negative market volatility risk premium. Carr and Wu (2009) calculated the variance premia for several stock market indexes through the replication with options. Average negative variance premium was shown.

The dynamics and driving forces of variance premium are studied in recent literature. Vilkov (2008) used the synthetic variance swap returns to approximate the variance risk premium and studied the dynamics and cross-sectional properties of variance premia embedded in index options and individual stock options. Todorov (2009) studied variance risk in terms of stochastic volatility and jumps. Model-free realized variance and realized jumps are constructed using high-frequency data. He found that price jumps play an important role in explaining the variance risk premium. Specifically, the estimated variance risk premium increases after a big market jump and slowly reverts to its long-run mean thereafter. Eraker (2008) captured the volatility premium and the large negative correlation between shocks to volatility and stock price with a general equilibrium based on long-run risk.

In this paper, we investigate whether the GARCH option pricing model can capture the variance premium. Since the seminal autoregressive conditional heteroscedasticity(ARCH) model of Engle (1982) and the generalized autoregressive conditional heteroscedasticity(GARCH) model of Bollerslev (1986), the GARCH model have attracted huge attention from the academics and practitioners and have been intensively used to model the financial times series. This is mostly because they can capture the volatility clustering and fat tails which are typical properties of the financial time series. The family of GARCH model has also been enriched to capture the stylized fact that the negative returns have higher impact on the volatility than the positive ones, which is called leverage effect. This class of GARCH models includes the exponential GARCH (EGARCH) of Nelson (1991), the threshold GARCH (TGARCH) of Glosten, Jagannathan and Runkle (1993) and the non-linear asymmetric GARCH (AGARCH) of Engle and Ng (1993) and the like. Engle and Lee (1993) introduced the component GARCH (CGARCH) which separates the conditional variance into a transitory component and a permanent component.

Duan (1995) pioneered in employing the GARCH model in the option pricing theory. He put forward an equilibrium argument that the options can be priced under a locally risk-neutral valuation relationship(LRNVR) with some assumption on the utility function when the price of the underlying asset follows a GARCH process. Kallsen and Taqqu (1998) considered a broad class of ARCH-type models embedded into a continuous-time framework and derived the same result by a no-arbitrage argument. The GARCH option pricing model has some linage with those bivariate diffusion option pricing models. Duan (1996,1997) showed that most variants of GARCH model mentioned above converge to the bivariate diffusion processes commonly used for modeling the stochastic volatility. Ritchken and Trevor (1999) developed a lattice algorithm that is applicable for option pricing under both GARCH models and bivariate diffusions. We will further discuss this limiting property in this paper. We find that this limiting process is somewhat pseudo in the sense that the diffusion limit of GARCH models is not identical with the true bivariate diffusion process.

To study the variance premium captured by GARCH option pricing model, we first calculate the GARCH implied VIX. VIX is the CBOE listed volatility index, which is updated in 2004 and reflects the expectation of the volatility of S&P 500 index over the next 30 calendar days. Demeterfi et al. (1999) showed that the squared VIX is actually the variance swap rate and can be replicated with portfolio of options written on S&P 500 index. In this paper, GARCH(1,1) and four other variants of GARCH(1,1) model are proposed for daily log-return of S&P 500 index. we then calculate the (squared)VIX as the risk-neutral expectation of the average variance over the next 22 trading days under the LRNVR proposed by Duan (1995). The VIX formulas for the five different GARCH models are derived.

We then compare the GARCH implied VIX with the CBOE VIX. We find that the GARCH implied VIX is significantly and consistently lower than the CBOE VIX for all the GARCH models investigated. The difference is around 3.6 which is consistent with the empirical evidence of the size of volatility premium. Thus, the GARCH option pricing model nearly does not capture any volatility premium or variance premium.

The source of this kind of incompleteness is then theoretically analyzed. Three arguments regarding to the diffusion limit of GARCH model and the change of probability measure from physical measure to locally risk neutral valuation relationship are given. It is shown that the framework of GARCH option pricing model fails to incorporate the price of volatility risk or variance premium. Thus, a new kind of risk neutral measure for GARCH option pricing model is called for.

The paper is constructed as follows. In Section 1 we review the change of measure for GARCH(p,q) model from the physical measure to the LRNVR of Duan (1995). In Section 2 we first propose the GARCH(1,1) for modeling S&P 500 index and derive out the VIX formula under LRNVR. Then we extend the linear GARCH(1,1) to other variants of GARCH(1,1) model including EGARCH, TGARCH, AGARCH and CGARCH, and give the VIX formulas, respectively. In Section 3 we estimate these GARCH models using the time series data of S&P 500 and compute the GARCH implied VIX. A comparison of the GARCH implied VIX and the CBOE VIX is then conducted. In Section 4 we theoretically investigate the locally risk neutral valuation relationship and the diffusion limit of GARCH model. The failure of the GARCH option pricing model to incorporate the price of volatility risk is analyzed. In Section 5 we conclude.

2 Theoretical Results on GARCH Implied VIX

2.1 GARCH Option Pricing Models

Duan (1995) utilized a linear GARCH process for modeling the underlying asset and pricing the options written on it. In that paper, the return of the asset in each period is modeled to follow a conditional lognormal distribution under the physical measure P,

$$\ln(\frac{X_t}{X_{t-1}}) = r + \lambda\sqrt{h_t} - \frac{1}{2}h_t + \epsilon_t, \tag{1}$$

where X_t is the price of the asset, r is constant interest rate and λ is the risk premium; ϵ_t follows a GARCH(p, q) process

$$\epsilon_t \mid \phi_{t-1} \sim N(0, h_t) \quad \text{under measure } P,$$
$$h_t = \alpha_0 + \sum_{i=1}^{q} \alpha_i \epsilon_{t-i}^2 + \sum_{i=1}^{p} \beta_i h_{t-i}, \tag{2}$$

With the assumptions made on utility function and aggregated consumption growth, Duan (1995) proposed a new locally risk neutral valuation relationship, Q, under which

$$\ln(\frac{X_t}{X_{t-1}}) = r - \frac{1}{2}h_t + \xi_t, \tag{3}$$

and

$$\xi_t \mid \phi_{t-1} \sim N(0, h_t),$$
$$h_t = \alpha_0 + \sum_{i=1}^{q} \alpha_i (\xi_{t-i} - \lambda\sqrt{h_{t-i}})^2 + \sum_{i=1}^{p} \beta_i h_{t-i}$$

2.2 GARCH Implied VIX

VIX reflects investors' expectation of the volatility of S&P 500 in the following 30 calender days, that is

$$(\frac{VIX_t}{100})^2 = E_t^Q[\frac{1}{\tau_0}\int_t^{t+\tau_0} \tilde{h}_s \, ds] \tag{4}$$

where $\tau_0 = 30$ and $\tilde{h}_s$ is the instantaneous annualized variance of the rate of return of S&P 500. In this paper, we calculate VIX as a expected arithmetic average of the variance in the n subperiods of the following 30 calendar days. That is

$$(\frac{VIX_t}{100})^2 = \frac{1}{n}\sum_{k=1}^{n} E_t^Q[\tilde{h}_{t+\frac{\tau_0 k}{n}}] \tag{5}$$

Especially, we will use data with daily frequency, that is $\tau_0 = n$, then

$$V_t = \frac{1}{n}\sum_{k=1}^{n} E_t^Q[h_{t+k}] \tag{6}$$

where $V_t = \frac{1}{252}(\frac{VIX_t}{100})^2$ is defined as a proxy for VIX_t in terms of daily variance.

The conditional mean of future variance can be calculated under various GARCH models. We now consider the GARCH(1,1), EGARCH(1,1), TGARCH(1,1), AGARCH(1,1) and Component GARCH(1,1) models and give their corresponding VIX formulas.

Proposition 1 *If S&P 500 follows a GARCH(1,1) process, then under the locally risk neutral valuation relationship Q proposed by Duan (1995), the implied VIX is a linear function of the conditional variance of the next period,*

$$V_t = A + Bh_{t+1} \tag{7}$$

where

$$A = \frac{\alpha_0}{1-\eta}(1-B),$$
$$B = \frac{1-\eta^n}{n(1-\eta)},$$
$$\eta = \alpha_1(1+\lambda^2) + \beta_1.$$

See the appendix for proof.

Given the process that the conditional variance follows, we know that a linear transformation of V_t will follow a stochastic process under the physical measure,

$$\tilde{V}_t = \alpha_0 + \alpha_1 \epsilon_t^2 + \beta_1 \tilde{V}_{t-1}, \tag{8}$$

and

$$\epsilon_t \mid \phi_{t-1} \sim N(0, \tilde{V}_{t-1}) \quad \text{under measure } P,$$

where $\tilde{V}_t = (V_t - A)/B$. We define $\tilde{V}_t = f(VIX_t)$ as a standardizing function.

Consider the EGARCH(1,1) mdoel:

$$\ln h_t = \alpha_0 + \beta_1 \ln h_{t-1} + g(z_{t-1}), \tag{9}$$

$$g(z_{t-1}) = \alpha_1 z_{t-1} + \delta(|z_{t-1}| - \sqrt{2/\pi}). \tag{10}$$

where $z_t = \epsilon_t / \sqrt{h_t}$ and the covariance stationary condition is $|\beta_1| < 1$. Following the GARCH option pricing model in Duan (1995), we can apply the locally risk neutral valuation relationship onto the EGARCH(1,1) model,

$$\ln h_t = \alpha_0 + \beta_1 \ln h_{t-1} + g(u_{t-1} - \lambda), \tag{11}$$

$$g(u_{t-1} - \lambda) = \alpha_1 (u_{t-1} - \lambda) + \delta(|u_{t-1} - \lambda| - \sqrt{2/\pi}). \tag{12}$$

where $u_t = \xi_t / \sqrt{h_t}$.

Proposition 2 *If S&P 500 follows an EGARCH(1,1) process, then under the locally risk neutral valuation relationship Q, the implied VIX formula takes the form of*

$$V_t = \frac{1}{n} h_{t+1} (1 + \sum_{j=2}^{n} \prod_{i=0}^{j-2} l_i) \tag{13}$$

with

$$\begin{aligned} l_i = e^{\beta_1^i (\alpha_0 - \delta\sqrt{2/\pi})} \Big\{ & e^{-\beta_1^i(\alpha_1 - \delta)\lambda + \frac{[\beta_1^i(\alpha_1-\delta)]^2}{2}} N\big[\lambda - \beta_1^i(\alpha_1 - \delta)\big] \\ & + e^{-\beta_1^i(\alpha_1 + \delta)\lambda + \frac{[\beta_1^i(\alpha_1+\delta)]^2}{2}} N\big[\beta_1^i(\alpha_1 + \delta) - \lambda\big] \Big\}. \end{aligned} \tag{14}$$

The proof is in Appendix.

Consider the TGARCH(1,1) mdoel:

$$h_t = \alpha_0 + \alpha_1 \epsilon_{t-1}^2 + \theta \epsilon_{t-1}^2 1(\epsilon_{t-1} < 0) + \beta_1 h_{t-1} \tag{15}$$

where $\alpha_0 > 0, \alpha_1 \geq 0$, and $\beta_1 > 0$. The covariance stationary condition is $\alpha_1 + \beta_1 + \frac{\theta}{2} < 1$. Under the locally risk neutral valuation relationship Q proposed by Duan (1995), the TGARCH(1,1) changes into

$$h_t = \alpha_0 + \alpha_1(\xi_{t-1} - \lambda\sqrt{h_{t-1}})^2 + \theta(\xi_{t-1} - \lambda\sqrt{h_{t-1}})^2 1(\xi_{t-1} - \lambda\sqrt{h_{t-1}} < 0) + \beta_1 h_{t-1} \tag{16}$$

Proposition 3 *If S&P 500 follows a TGARCH(1,1) process, then under the locally risk neutral valuation relationship Q, the implied VIX formula takes the form of*

$$V_t = C + D h_{t+1} \tag{17}$$

where

$$C = \frac{\alpha_0}{1-\zeta}(1-D),$$
$$D = \frac{1-\zeta^n}{n(1-\zeta)},$$
$$\zeta = \alpha_1(1+\lambda^2) + \beta_1 + \theta\big[\frac{\lambda}{\sqrt{2\pi}} e^{-\frac{\lambda^2}{2}} + (1+\lambda^2)N(\lambda)\big].$$

See Appendix for details.

Consider the AGARCH(1,1) mdoel:

$$h_t = \alpha_0 + \alpha_1(\epsilon_{t-1} - \theta\sqrt{h_{t-1}})^2 + \beta_1 h_{t-1} \tag{18}$$

where $\alpha_0 > 0, \alpha_1 \geq 0$, and $\beta_1 > 0$. Under the locally risk neutral valuation relationship Q proposed by Duan (1995), the AGARCH(1,1) changes into

$$h_t = \alpha_0 + \alpha_1(\xi_{t-1} - \lambda\sqrt{h_{t-1}} - \theta\sqrt{h_{t-1}})^2 + \beta_1 h_{t-1} \tag{19}$$

Proposition 4 *If S&P 500 follows an AGARCH(1,1) process, then under the locally risk neutral valuation relationship Q, the implied VIX formula takes the form of*

$$V_t = E + F h_{t+1} \tag{20}$$

where

$$E = \frac{\alpha_0}{1-\zeta}(1-F),$$
$$F = \frac{1-\zeta^n}{n(1-\zeta)},$$
$$\zeta = \alpha_1\left[1+(\lambda+\theta)^2\right]+\beta_1.$$

See Appendix for details.

Engle and Lee (1993) extended the GARCH model to the Component GARCH (CGARCH) model by decomposing the conditional variance into a transitory component and a permanent component. Consider the component GARCH(1,1) model:

$$h_t - q_t = \alpha_1(\epsilon_{t-1}^2 - q_{t-1}) + \beta_1(h_{t-1} - q_{t-1}), \tag{21}$$
$$q_t = \alpha_0 + \rho q_{t-1} + \phi(\epsilon_{t-1}^2 - h_{t-1}). \tag{22}$$

where $\alpha_0 > 0, \alpha_1 0, \beta_1 > \phi > 0$, and $1 > \rho > \alpha_1 + \beta_1 > 0$. In CGARCH model, $h_t - q_t$ is the transitory component which shrinks to zero and q_t is the permanent component which converges to $\alpha_0/(1-\rho)$. Under the locally risk neutral valuation relationship Q proposed by Duan (1995), the CGARCH(1,1) changes into

$$h_t - q_t = \alpha_1[(\xi_{t-1} - \lambda\sqrt{h_{t-1}})^2 - q_{t-1}] + \beta_1(h_{t-1} - q_{t-1}), \tag{23}$$
$$q_t = \alpha_0 + \rho q_{t-1} + \phi[(\xi_{t-1} - \lambda\sqrt{h_{t-1}})^2 - h_{t-1}]. \tag{24}$$

Proposition 5 *If S&P 500 follows a CGARCH(1,1) process, then under the locally risk neutral valuation relationship Q, the implied VIX can be computed as a linear function of the transitory and permanent components of the conditional variance of next period,* h_{t+1} *and* q_{t+1}.

See Appendix for proof.

3 Data and Estimation

In this section, we use maximum likelihood method to estimate the GARCH model in the physical measure with the time series data of the close price of S&P 500 index. The sample period ranges from Jan 2, 1990 to Aug 10, 2009. We also use the daily 3-month U.S. Treasury bills(secondary market) rate as the risk-free rate and we get this time series data from the Federal Reserve web site.

When running the maximum likelihood estimations, we set the initial conditional variance as the variance of the rate of return of S&P 500 index over the whole time period.

4 Results

With the estimates from maximum likelihood estimations, we can figure out the time series of conditional variance of S&P 500 index under the GARCH(1,1) model. We then can calculate the time series of VIX under GARCH model with the GARCH implied VIX formulas worked out in Section 2. Comparisons of the GARCH implied VIX and CBOE VIX are conducted for all the five versions of GARCH(1,1) models, and are shown in Figure 1 to Figure 5. The statistics of the GARCH implied VIX, CBOE VIX and their differences are shown in Table 1. Generally speaking, the derived VIX is highly correlated with the real VIX, which can seen from the trends in the figures. The correlation coefficients between the GARCH implied VIX and CBOE VIX are very high, ranging from 0.92 to 0.94 for different GARCH models. However, the VIX time series derived from the GARCH model are lower than the real VIX data. The null hypothesis that the GARCH implied VIX has the same mean with the CBOE VIX is strongly rejected as the P-value in Table 1 is very low. The mean errors(CBOE VIX minus GARCH implied VIX) are very close among the five GARCH model, with the minimum 3.47 of the AGARCH model and maximum 3.78 of TGARCH model. The EGARCH model gains the minimum mean absolute error of 3.76, and the TGARCH model has a maximum of 4.08. In terms of root mean squared error, AGARCH model gets the lowest value of 4.73 and TGARCH model performs worst with a value of 4.98.

It is important to note that the magnitude of the mean error of between CBOE VIX and GARCH implied VIX, ranging from 3.47 to 3.78, is nearly consistent with that of variance premium in standard deviation unit, which is about 3.3. Thus, the GARCH implied VIX undervalues the CBOE VIX by a amount similar to the variance premium.

5 Discussion

The empirical results show that the GARCH option pricing model can not capture the variance premium, which indicates the incompleteness of the model. We will illustrate this point with the case of AGARCH(1,1) model. Under the physical probability measure P, the model is specified as

$$\ln(\frac{X_t}{X_{t-1}}) = r + \lambda\sqrt{h_t} - \frac{1}{2}h_t + \sqrt{h_t}\upsilon_t, \tag{25}$$

$$h_t = \alpha_0 + \alpha_1 h_{t-1}(\upsilon_{t-1} - \theta)^2 + \beta_1 h_{t-1}, \tag{26}$$

where υ_t is a standard normal random variable, conditional on the information at time t-1; $\alpha_0 > 0, \alpha_1 \geq 0, \beta_1 \geq 0$ and $(1+\theta^2)\alpha_1 - \beta_1 < 1$ for a covariance staionary process. A positive θ can capture the negative correlation between the return and conditional variance as

$$Cov^P(\upsilon_t, h_{t+1}) = -2\theta\alpha_0\alpha_1[1-(1+\theta^2)\alpha_1-\beta_1]^{-1}. \tag{27}$$

If $\theta = 0$, this AGARCH model degenerates to a Linear GARCH(1,1) model discussed in Duan (1995), where the return and conditional variance are not correlated.

Under the LRNVR Q, the prices seem to evolve in a risk neutral world

$$\ln(\frac{X_t}{X_{t-1}}) = r - \frac{1}{2}h_t + \sqrt{h_t}\varepsilon_t, \tag{28}$$

$$h_t = \alpha_0 + \alpha_1 h_{t-1}(\varepsilon_{t-1}-\theta^*)^2 + \beta_1 h_{t-1}, \tag{29}$$

where ε_t is a standard normal random variable under LRNVR Q, conditional on the information at time t-1, and $\theta^* = \theta + \lambda$.

Duan (1996,1997) studied the diffusion limit of the GARCH model. Divide each time period(”day”) into n sub-period of width $\Delta t = 1/n$. For k=1,2,...,n, approximating process is constructed as

$$\ln(X_{ks}^{(n)}) = \ln(X_{(t-1)s}^{(n)}) + (r + \lambda\sqrt{h_{ks}^{(n)}} - \frac{1}{2}h_{ks}^{(n)})\Delta t + \sqrt{h_{ks}^{(n)}}\sqrt{\Delta t}\upsilon_k, \tag{30}$$

$$h_{(k+1)s}^{(n)} - h_{ks}^{(n)} = \alpha_0\Delta t + h_{ks}^{(n)}[\alpha_1 q + \beta_1 - 1]\Delta t + h_{ks}^{(n)}\alpha_1\sqrt{\Delta t}[(\upsilon_k-\theta)^2 - q], \tag{31}$$

where $\upsilon_k, k = 1, 2, \ldots$ is a sequence of i.i.d standard normal random variables; $q = 1+\theta^2$. And the corresponding process under the LRNVR Q is

$$\ln(X_{ks}^{(n)}) = \ln(X_{(t-1)s}^{(n)}) + (r - \frac{1}{2}h_{ks}^{(n)})\Delta t + \sqrt{h_{ks}^{(n)}}\sqrt{\Delta t}\varepsilon_k, \tag{32}$$

$$\begin{aligned} h_{(k+1)s}^{(n)} - h_{ks}^{(n)} &= \alpha_0\Delta t + h_{ks}^{(n)}[\alpha_1 q + \beta_1 - 1]\Delta t \\ &\quad + h_{ks}^{(n)}\alpha_1\sqrt{\Delta t}[(\varepsilon_k - \theta - \lambda\sqrt{\Delta t})^2 - q], \end{aligned} \tag{33}$$

where $\varepsilon_k = \upsilon_k + \lambda\sqrt{\Delta t}, k = 1, 2, \ldots$ is a sequence of i.i.d standard normal random variables under Q.

Duan shows that the limiting diffusion process of the approximating process (33) and

(34) of AGARCH(1,1) under the physical measure P is

$$d\ln(X_t) = (r + \lambda\sqrt{h_t} - \frac{1}{2}h_t)dt + \sqrt{h_t}dW_{1t}, \tag{34}$$

$$dh_t = [\alpha_0 + (\alpha_1 q + \beta_1 - 1)h_t]dt - 2\theta\alpha_1 h_t dW_{1t} + \sqrt{2}\alpha_1 h_t dW_{2t}, \tag{35}$$

where dW_{1t} and dW_{2t} are independent Wiener processes. And the limiting diffusion process of the approximating process (35) and (36) of AGARCH(1,1) under the LRNVR Q is

$$d\ln(X_t) = (r - \frac{1}{2}h_t)dt + \sqrt{h_t}dZ_{1t}, \tag{36}$$

$$dh_t = [\alpha_0 + (\alpha_1 q + \beta_1 - 1 + 2\lambda\alpha_1\theta)h_t]dt - 2\theta\alpha_1 h_t dZ_{1t} + \sqrt{2}\alpha_1 h_t dZ_{2t}, \tag{37}$$

where dZ_{1t} and dZ_{2t} are independent Wiener processes. Particularly, $dZ_{1t} = dW_{1t} + \lambda dt, dZ_{2t} = dW_{2t}$.

We now put forward three arguments about the locally risk neutral valuation relationship proposed by Duan(1995) and the diffusion limit properties of the GARCH model.

Firstly, the diffusion limit no matter under the physical measure or the LRNVR is not identical with the traditionally mentioned "true" bivariate diffusion processes such as Wiggins (1987), Johnson and Shanno (1987), Hull and White (1987), Scott (1987) and Heston (1993). In these models, the innovations in price and volatility are totally independent. However, the innovations in price and volatility in the limiting diffusion process of GARCH model still have kinds of linkage though being statistically independent. Under the physical measure, the item $\sqrt{\Delta t}\upsilon_k$ in the approximating process of (33) converges to the Wiener process of dW_{1t}. The last item on the right hand of the approximating process of (34) can be decomposed as

$$\sqrt{\Delta t}[(\upsilon_k - \theta)^2 - q] = -2\theta\sqrt{\Delta t}\upsilon_k + \sqrt{\Delta t}(\upsilon_k^2 - 1). \tag{38}$$

Comparing it with its diffusion limit in (38), we find that the innovation in the volatility process, $\sqrt{2}dW_{2t}$, is actually the limit of $\sqrt{\Delta t}(\upsilon_k^2 - 1)$, noting that $\upsilon_k^2 - 1$ has mean zero and variance of 2. However, although $\upsilon_k^2 - 1$ is uncorrelated with υ_k, they still have linkage through higher moments. For example, $E^P[\upsilon_k^2 \cdot (\upsilon_k^2 - 1)] = 1$. Thus, the relationship between dW_{1t} and dW_{2t} are not the same as that in the common bivariate diffusion models.

Under the LRNVR Q, the item $\sqrt{\Delta t}\varepsilon_k$ in the approximating process of (35) converges to the Wiener process of dZ_{1t}. The last item on the right hand of the approximating process of (36) can be decomposed as

$$\sqrt{\Delta t}[(\varepsilon_k - \theta - \lambda\sqrt{\Delta t})^2 - q] = 2\theta\lambda\Delta t - 2\theta\sqrt{\Delta t}\varepsilon_k + \sqrt{\Delta t}(\varepsilon_k^2 - 1) + \lambda^2 \sqrt[3]{(\Delta t)^2} - 2\lambda\Delta t\varepsilon_k. \quad (39)$$

As Δt tends to zero, $\sqrt{\Delta t}(\varepsilon_k^2 - 1)$ converges to dZ_{2t}, and the last two items disappear according to Ito's Lemma. For the same reason above, dZ_{1t} and dZ_{2t} are linked though statistically independent.

Secondly, for the true bivariate diffusion model, a price of volatility risk is usually introduced to the volatility process when we move from the physical measure to the risk neutral measure. This is because volatility has its own risk which has to be compensated. However, the diffusion limit of GARCH model under LRNVR Q does not reflect this kind of compensation for volatility risk. Actually, Under the probability measure change from physical measure to LRNVR Q, the innovation in volatility process is invariant, $dZ_{2t} = dW_{2t}$, as

$$\begin{aligned}\sqrt{\Delta t}(\varepsilon_k^2 - 1) &= \sqrt{\Delta t}[(\upsilon_k + \lambda\sqrt{\Delta t})^2 - 1] \\ &= \sqrt{\Delta t}(\upsilon_k^2 - 1) + 2\lambda\upsilon_k\Delta t + \lambda^2 \sqrt[3]{(\Delta t)^2}.\end{aligned} \quad (40)$$

The left hand converges to dZ_{2t} and the right hand converges to dW_{2t}. The failure of the diffusion limit to incorporate the price of volatility risk resulted from the inability of the GARCH option pricing model to account for the volatility risk.

Thirdly, the presence of the equity premium λ under the LRNVR Q in the volatility process of both the GARCH model (34) and its diffusion limit (40) does not represent the incorporation of variance premium. Consider the linear GARCH(1,1) with $\theta = 0$. As shown in the diffusion limit of the volatility process (40)under LRNVR, the item containing equity premium λ will disappear. Some may argue that the variance premium derives from the negative correlation between the price and volatility, and the absence of variance premium in linear GARCH is caused since the price and volatility are not correlated as shown in (30). However, even the price process and the volatility process are not correlated in the bivariate diffusions, the price of volatility risk will still show up under the risk neutral measure. Moreover, literatures show that very little of the volatility risk premium can be explained by the market risk or the correlation of volatility with prices. Instead, It may be driven by some other risk factors including jump risk. Thus, the equity risk premium does not simultaneously represent the volatility risk premium.

By now we have theoretically demonstrated that the GARCH option pricing model under locally risk neutral valuation relationship is not capable of capturing the variance premium. This suggests the incompleteness of the locally risk neutral valuation relationship now widely used in GARCH option pricing literatures.

6 Conclusion

In this paper, we follow the GARCH option pricing model of Duan (1995) and calculate the VIX squared as the expected arithmetic average of the conditional variance over the next 22 trading days under the locally risk neutral valuation relationship. GARCH implied VIX formulas are derived for linear GARCH(1,1) and four other extensions of GARCH(1,1) models.

We use the time series of the close price of S&P 500 index since the launch of VIX on CBOE to run the maximum likelihood estimation of the GARCH models. The corresponding VIX time series are then calculated. The comparison of these GARCH impied VIX with the CBOE VIX shows that the GARCH implied VIX is significantly and consistently lower than the CBOE VIX for all kinds of GARCH model investigated. Moreover, the magnitude of the difference is coincident with the empirical variance premium. This indicates that the GARCH option pricing under LRNVR can not capture the variance premium.

With the case of AGARCH(1,1), we illustrate the reasons that the GARCH option pricing model fails to incorporate the price of volatility risk compared with the bivariate diffusion models. Firstly, the diffusion limit of the GARCH model is somewhat tricky and actually different from the bivariate diffusions. Secondly, the innovation of volatility is invariant with respect to probability measure change from the physical measure to the LRNVR. Finally, we point out that the equity risk premium can not serve to capture the variance premium, which is usually misunderstood in literatures.

The empirical results and the theoretical arguments both indicate that the GARCH option pricing model under locally risk neutral valuation relationship is not capable of capturing the variance premium. This suggests that the locally risk neutral valuation relationship is not complete and kind of fully risk neutral measure for GARCH option pricing is called for.

Appendix

Proof of Proposition 1. For $k > 1$,

$$\begin{aligned} E_t^Q(h_{t+k}) &= E_t^Q[\alpha_0 + \alpha_1(\xi_{t+k-1} - \lambda\sqrt{h_{t+k-1}})^2 + \beta_1 h_{t+k-1}] \\ &= E_t^Q\{\alpha_0 + E_{t+k-2}^Q[\alpha_1 h_{t+k-1}(\frac{\xi_{t+k-1}}{\sqrt{h_{t+k-1}}} - \lambda)^2 + \beta_1 h_{t+k-1}]\} \\ &= E_t^Q\{\alpha_0 + h_{t+k-1}[\alpha_1(1+\lambda^2) + \beta_1]\} \\ &= \alpha_0 + [\alpha_1(1+\lambda^2) + \beta_1]E_t^Q(h_{t+k-1}), \end{aligned} \tag{41}$$

and continuing this iterating process, we have

$$E_t^Q(h_{t+k}) = \alpha_0 \sum_{i=0}^{k-2}[\alpha_1(1+\lambda^2) + \beta_1]^i + [\alpha_1(1+\lambda^2) + \beta_1]^{k-1} h_{t+1}. \tag{42}$$

When the forecasting horizon goes to infinity, the conditional expected variance will converge to the unconditional expected variance, $\alpha_0/(1 - \alpha_1(1+\lambda^2) - \beta_1)$, and the effect of the present conditional variance will shrink out.

Substituting this result into equation (6), we get the VIX_t as a linear function of the conditional variance of the next period,

$$V_t = A + Bh_{t+1} \tag{43}$$

where

$$\begin{aligned} A &= \frac{\alpha_0}{1-\eta}(1-B), \\ B &= \frac{1-\eta^n}{n(1-\eta)}, \\ \eta &= \alpha_1(1+\lambda^2) + \beta_1. \end{aligned}$$

Proof of Proposition 2. Under the LRNVR Q, the expectation of the conditional variance k periods ahead can be expressed as

$$
\begin{aligned}
E_t^Q(h_{t+k}) &= e^{\alpha_0-\delta\sqrt{2/\pi}} E_t^Q(h_{t+k-1}^{\beta_1}) E_{t+k-2}^Q(e^{\alpha_1(u_{t+k-1}-\lambda)+\delta|u_{t+k-1}-\lambda|}) \\
&= e^{\alpha_0-\delta\sqrt{2/\pi}} E_t^Q(h_{t+k-1}^{\beta_1}) \Big(\int_{-\infty}^{\lambda} e^{(\alpha_1-\delta)(u_{t+k-1}-\lambda)} \frac{1}{\sqrt{2\pi}} e^{-\frac{u_{t+k-1}^2}{2}} du_{t+k-1} \\
&\quad + \int_{\lambda}^{\infty} e^{(\alpha_1+\delta)(u_{t+k-1}-\lambda)} \frac{1}{\sqrt{2\pi}} e^{-\frac{u_{t+k-1}^2}{2}} du_{t+k-1} \Big) \\
&= e^{(\alpha_0-\delta\sqrt{2/\pi})} \big[e^{-(\alpha_1-\delta)\lambda+\frac{(\alpha_1-\delta)^2}{2}} N(\lambda-\alpha_1+\delta) \\
&\quad + e^{-(\alpha_1+\delta)\lambda+\frac{(\alpha_1+\delta)^2}{2}} N(\alpha_1+\delta-\lambda) \big] E_t^Q(h_{t+k-1}^{\beta_1}).
\end{aligned} \tag{44}
$$

For $0 \le i \le k-2$, we have

$$
\beta_1^i \ln h_{t+k-i} = \beta_1^i(\alpha_0-\delta\sqrt{2/\pi}) + \beta_1^{i+1} \ln h_{t+k-i-1} + \beta_1^i \big[\alpha_1(u_{t+k-i-1}-\lambda) + \delta|u_{t+k-i-1}-\lambda| \big] \tag{45}
$$

Thus,

$$
\begin{aligned}
E_t^Q(h_{t+k-i}^{\beta_1^i}) &= e^{\beta_1^i(\alpha_0-\delta\sqrt{2/\pi})} E_t^Q\big(h_{t+k-i-1}^{\beta_1^{i+1}}\big) E_{t+k-i-2}^Q\big\{ e^{\beta_1^i[\alpha_1(u_{t+k-i-1}-\lambda)+\delta|u_{t+k-i-1}-\lambda|]} \big\} \\
&= e^{\beta_1^i(\alpha_0-\delta\sqrt{2/\pi})} E_t^Q\big(h_{t+k-i-1}^{\beta_1^{i+1}}\big) \Big(\int_{-\infty}^{\lambda} e^{\beta_1^i(\alpha_1-\delta)(u_{t+k-i-1}-\lambda)} \frac{1}{\sqrt{2\pi}} e^{-\frac{u_{t+k-i-1}^2}{2}} du_{t+k-i-1} \\
&\quad + \int_{\lambda}^{\infty} e^{\beta_1^i(\alpha_1+\delta)(u_{t+k-i-1}-\lambda)} \frac{1}{\sqrt{2\pi}} e^{-\frac{u_{t+k-i-1}^2}{2}} du_{t+k-i-1} \Big) \\
&= e^{\beta_1^i(\alpha_0-\delta\sqrt{2/\pi})} \Big\{ e^{-\beta_1^i(\alpha_1-\delta)\lambda+\frac{[\beta_1^i(\alpha_1-\delta)]^2}{2}} N\big[\lambda-\beta_1^i(\alpha_1-\delta)\big] \\
&\quad + e^{-\beta_1^i(\alpha_1+\delta)\lambda+\frac{[\beta_1^i(\alpha_1+\delta)]^2}{2}} N\big[\beta_1^i(\alpha_1+\delta)-\lambda\big] \Big\} E_t^Q\big(h_{t+k-i-1}^{\beta_1^{i+1}}\big).
\end{aligned} \tag{46}
$$

Denote

$$
\begin{aligned}
l_i = e^{\beta_1^i(\alpha_0-\delta\sqrt{2/\pi})} \Big\{ & e^{-\beta_1^i(\alpha_1-\delta)\lambda+\frac{[\beta_1^i(\alpha_1-\delta)]^2}{2}} N\big[\lambda-\beta_1^i(\alpha_1-\delta)\big] \\
& + e^{-\beta_1^i(\alpha_1+\delta)\lambda+\frac{[\beta_1^i(\alpha_1+\delta)]^2}{2}} N\big[\beta_1^i(\alpha_1+\delta)-\lambda\big] \Big\}.
\end{aligned} \tag{47}
$$

Then starting from formula (44) and iterating with formula (46), we have

$$
E_t^Q(h_{t+k}) = \prod_{i=0}^{k-2} l_i h_{t+1}. \tag{48}
$$

And the VIX formula implied by EGARCH(1,1) is

$$V_t = \frac{1}{n} h_{t+1}(1 + \sum_{j=2}^{n} \prod_{i=0}^{j-2} l_i) \tag{49}$$

Proof of Proposition 3. Under the LRNVR Q, the expectation of the conditional variance k periods ahead can be expressed as

$$\begin{aligned} E_t^Q(h_{t+k}) &= E_t^Q\Big[\alpha_0 + \alpha_1(\xi_{t+k-1} - \lambda\sqrt{h_{t+k-1}})^2 + \beta_1 h_{t+k-1} \\ &\quad + \theta(\xi_{t+k-1} - \lambda\sqrt{h_{t+k-1}})^2 1(\xi_{t+k-1} - \lambda\sqrt{h_{t+k-1}} < 0)\Big] \\ &= E_t^Q\Big\{\alpha_0 + E_{t+k-2}^Q\big[\alpha_1 h_{t+k-1}(\frac{\xi_{t+k-1}}{\sqrt{h_{t+k-1}}} - \lambda)^2 + \beta_1 h_{t+k-1} \\ &\quad + \theta h_{t+k-1}\int_{-\infty}^{\lambda}(u_{t+k-1} - \lambda)^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{u_{t+k-1}^2}{2}} du_{t+k-1}\big]\Big\} \\ &= \alpha_0 + \Big\{\alpha_1(1+\lambda^2) + \beta_1 + \theta\big[\frac{\lambda}{\sqrt{2\pi}} e^{-\frac{\lambda^2}{2}} + (1+\lambda^2)N(\lambda)\big]\Big\} E_t^Q(h_{t+k-1}), \end{aligned} \tag{50}$$

Then following the same procedure as that of GARCH(1,1), we can get the VIX formula for TGARCH(1,1).

Proof of Proposition 4. Under the LRNVR Q, the expectation of the conditional variance k periods ahead can be expressed as

$$\begin{aligned} E_t^Q(h_{t+k}) &= E_t^Q\Big\{\alpha_0 + E_{t+k-2}^Q\big[\alpha_1(\xi_{t-1} - \lambda\sqrt{h_{t-1}} - \theta\sqrt{h_{t-1}})^2 + \beta_1 h_{t-1}\big]\Big\} \\ &= \alpha_0 + \big\{\alpha_1[1 + (\lambda+\theta)^2] + \beta_1\big\} E_t^Q(h_{t+k-1}). \end{aligned} \tag{51}$$

Then following the same procedure as that of GARCH(1,1), we can also get the VIX formula for AGARCH(1,1).

Proof of Proposition 5. Under the LRNVR Q, the expectations of the transitory and permanent components of the conditional variance k periods ahead can be calculated, respectively. For $k > 1$,

$$\begin{aligned} E_t^Q(h_{t+k}) &= E_t^Q\Big\{\alpha_0 + E_{t+k-2}^Q\big[\rho q_{t+k-1} + \phi[(\xi_{t+k-1} - \lambda\sqrt{h_{t+k-1}})^2 - h_{t+k-1}] \\ &\quad + \alpha_1[(\xi_{t+k-1} - \lambda\sqrt{h_{t+k-1}})^2 - q_{t+k-1}] + \beta_1(h_{t+k-1} - q_{t+k-1})\big]\Big\} \\ &= \alpha_0 + [\alpha + \beta + (\phi + \alpha)\lambda^2] E_t^Q(h_{t+k-1}) + (\rho - \alpha - \beta) E_t^Q(q_{t+k-1}), \end{aligned} \tag{52}$$

and

$$\begin{aligned} E_t^Q(q_{t+k}) &= E_t^Q\Big\{\alpha_0 + E_{t+k-2}^Q\big[\rho q_{t+k-1} + \phi[(\xi_{t+k-1} - \lambda\sqrt{h_{t+k-1}})^2 - h_{t+k-1}]\big]\Big\} \\ &= \alpha_0 + \phi\lambda^2 E_t^Q(h_{t+k-1}) + \rho E_t^Q(q_{t+k-1}). \end{aligned} \tag{53}$$

They can be expressed in vector form as

$$E_t^Q \begin{pmatrix} h_{t+k} \\ q_{t+k} \end{pmatrix} = \alpha_0 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} \alpha + \beta + (\phi + \alpha)\lambda^2 & \rho - \alpha - \beta \\ \phi\lambda^2 & \rho \end{pmatrix} E_t^Q \begin{pmatrix} h_{t+k-1} \\ q_{t+k-1} \end{pmatrix}. \tag{54}$$

Continuing this iterating process, It is straightforward that the expectation of the transitory component of the conditional variance k periods ahead, $E_t^Q(h_{t+k})$, can be computed as a linear function of the transitory and permanent components of the conditional variance of next period, h_{t+1} and q_{t+1}. Then the implied VIX under CGARCH(1,1), which is the average of the expectation of the conditional variance over the next 30 calendar days, can also be expressed as a linear function of h_{t+1} and q_{t+1}.

References

[1] Anderson, T.G., Bollerslev, T., Diebold, F.X. and Labys, P., 2003, Modeling and Forecasting Realized Volatility, *Econometrica* 71, 579-625.

[2] Bakshi, G., Kapadia, N., 2003, Delat-Hedged Gains and the Negative Market Volatility Risk Premium, *Review of Financial Studies* 16, 527-566.

[3] Bollerslev, T., 1986, Generalized Autoregressive Conditional Heteroskedasticity, *Journal of Econometrics* 31, 637-659.

[4] Bollerslev, T., Tanchen, G. and Zhou, H., 2008, Expected Stock Return and Variance Risk Premia, *Review of Financial Studies*, forthcoming.

[5] Carr, P., and Wu, L., 2009, Variance Risk Premium, *Review of Financial Studies* 22, 1311-41.

[6] Coval, J.D., and Shumway, T., 2001, Expected Option Returns, *Journal of Finance* 56, 983-1010.

[7] Duan, J.-C, 1995, The Garch Option Pricing Model, *Mathematical Finance* 5, 13-32.

[8] Duan, J.-C, 1996, A Unified Theory of Option Pricing under Stochastic Volatility-from GARCH to Diffusion, Unpublished manuscript, Hong Kong University of Science and Technology.

[9] Duan, J.-C, 1997, Augmented GARCH(p,q) Process and its Diffusion Limit, *Journal of Econometrics* 79, 97-127.

[10] Demeterfi, K., Derman, E., Kamal, M., Zou, J., 1999, A Guide to Volatility and Variance Swaps, *Journal of Derivatives* 6, 9-32.

[11] Engle, R., 1982, Autoregressive Conditional Heteroskedasticity with Estimates of the Variance of U.K. Inflation, *Econometrica* 50, 987-1008.

[12] Engle, R., and Lee, G., 1993, A Permanent and Transitory Component Model of Stock Return Volatility, UCSD, Department of Economics, Discussion Paper No: 92-44R.

[13] Engle, R., and Ng, V., 1993, Measuring and Testing the Impact of News on Volatility, *Journal of Finance* 48, 1749-1778.

[14] Eraker, B., 2008, The Voaltility Premium, Working Papre, Duke University.

[15] Glosten, L., Jagannathan, R., and Runkle, D., 1993, Relationship Between the Expected Value and the Volatility of the Nominal Excess Return on Stocks. *Journal of Finance* 48, 1779-1801.

[16] Heston, S., 1993, A Closed Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options, *Review of Financial Studies* 6, 327-343.

[17] Hull, J., and White, A., 1987, The Pricing of Options on Assets with Stochastic Volatility, *Journal of Finance* 42, 281-300.

[18] Johnson, H., and Shanno, D., 1987, Option Pricing When the Variance is Changing, *Journal of Financial Quantitative Analysis* 22, 143-151.

[19] Kallsen, J. and Taqqu, M.S., 1998, Option Pricing in ARCH-Type Models, *Mathematical Finance* 8, 13-26.

[20] Nelson, D., 1991, Conditional Heteroskedasticity in Asset Return: A New Approach, *Econometrica* 59, 347-370.

[21] Ritchken, P. and Trevor, R., 1999, Pricing Options under Generalized GARCH and Stochastic Volatility Processes, *Journal of Finance* 54, 377-402.

[22] Scott, L., 1987, Option Pricing When the Variance Changes Randomly: Theory, Estimation, and and Application, *Journal of Financial Quantitative Analysis* 22, 419-438.

[23] Todorov, V., 2009, Variance Risk-Premium Dynamics: The Role of Jumps, *Review of Financial Studies*, forthcoming.

[24] Vilkov, G., 2008, Variance Risk Premium Demystified, Working Paper, Boulevard de Constance.

[25] Wiggins, J., 1987, Option Values under Stochastic Volatitlity: Theoy and Empirical Estimates, *Journal of Finance* 19, 351-372.

Table 1: **Evaluation of the difference between GARCH implied VIX and CBOE VIX over the whole period**
This table shows the statistics of the difference between the GARCH implied VIX and CBOE VIX for the five versions of GARCH model investigated over the time period from Jan 2, 1990 to Aug 10, 2009. The error is calculated as CBOE VIX minus GARCH implied VIX. The mean error(ME) calculates the daily average error between the GARCH implied VIX and CBOE VIX. The standard error(Std.Err.) calculates the standard deviation of the error. The mean absolute error(MAE) calculates the daily average absolute error between the GARCH implied VIX and CBOE VIX. The mean squared error(MSE) calculates the daily average squared error between the GARCH implied VIX and CBOE VIX. The root mean squared error(RMSE) calculates the square root of the mean squared error. The P-value is for the null hypothesis that the means of GARCH implied VIX and CBOE VIX are equal. Violation of one-sigma band represent the probability that the GARCH implied VIX lies out of the one-standard-deviation band of the CBOE VIX. The correlation coefficient(Corr. Coef.) calculates the linear correlation between GARCH implied VIX and CBOE VIX

Model	GARCH	EGARCH	TGARCH	AGARCH	CGARCH
ME	3.63	3.62	3.78	3.47	3.66
Std.Err.	3.31	3.12	3.24	3.22	3.06
MAE	4.02	3.76	4.08	3.79	3.91
MSE	24.13	22.81	24.78	22.39	22.79
RMSE	4.91	4.78	4.98	4.73	4.77
P-value	0.00	0.00	0.00	0.00	0.00
Violation of one-sigma band	7.86%	8.08%	8.27%	7.44%	6.69%
Corr. Coef.	0.92	0.94	0.93	0.93	0.93

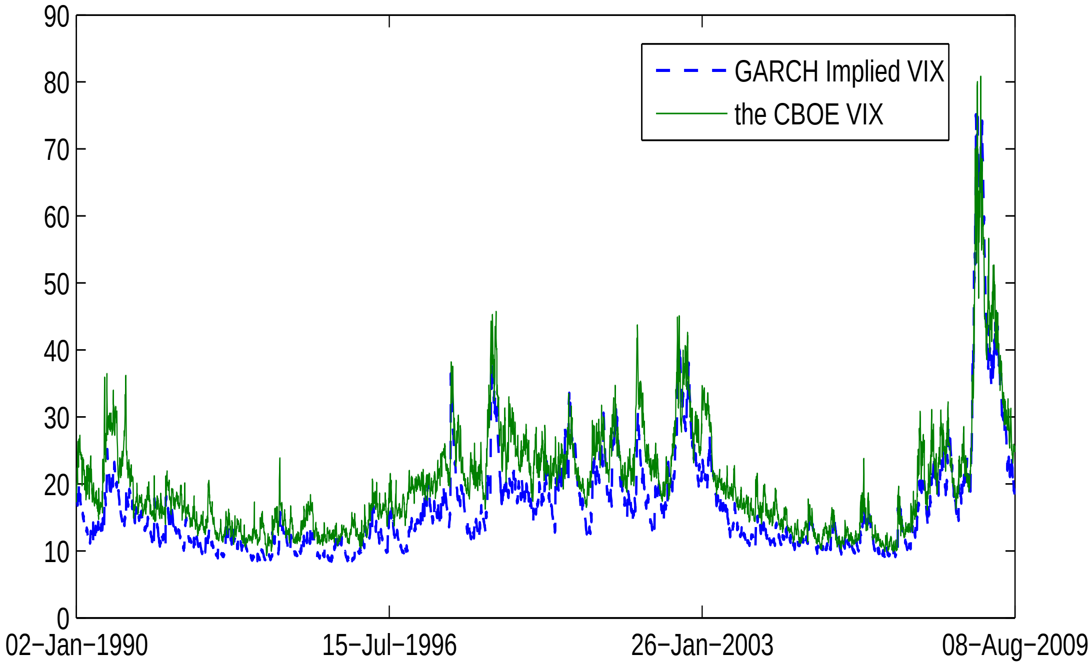

Figure 1: **The Comparison between GARCH Implied VIX and the CBOE VIX from Jan 2, 1990 to Aug 8, 2009**
The estimated GARCH(1,1) model is:
$h_t = 7.245 \times 10^{-7} + 0.06323\epsilon^2_{t-i} + 0.9312h_{t-i}$,
and estimated $\lambda = 0.05235$. The conditional variance of the first period are both set at the variance of the rate of return of S&P 500 over the whole sample period, which is 18.63% on annual basis.

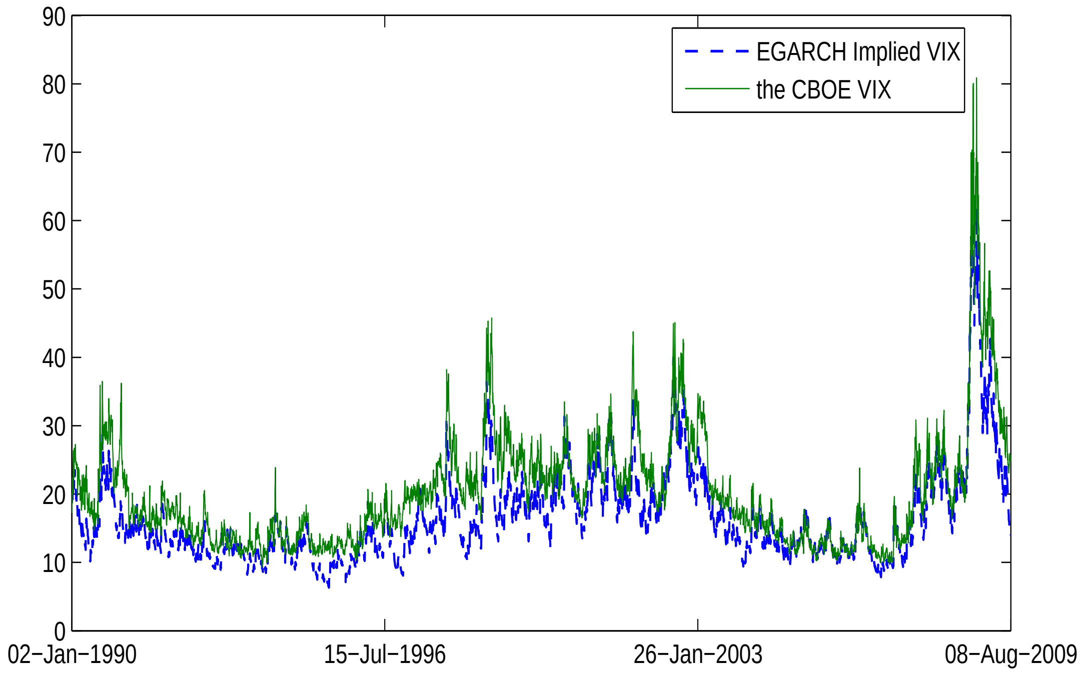

Figure 2: **The Comparison between EGARCH Implied VIX and the CBOE VIX from Jan 2, 1990 to Aug 8, 2009**
The estimated EGARCH(1,1) model is:
$\ln h_t = -0.1329 + 0.9854 \ln h_{t-1} + g(z_{t-1})$,
$g(z_{t-1}) = -0.09207 z_{t-1} + 0.1105(|z_{t-1}| - \sqrt{2/\pi})$,
and estimated $\lambda = 0.01675$. The conditional variance of the first period is set at the variance of the rate of return of S&P 500 over the whole sample period, which is 18.63% on annual basis.

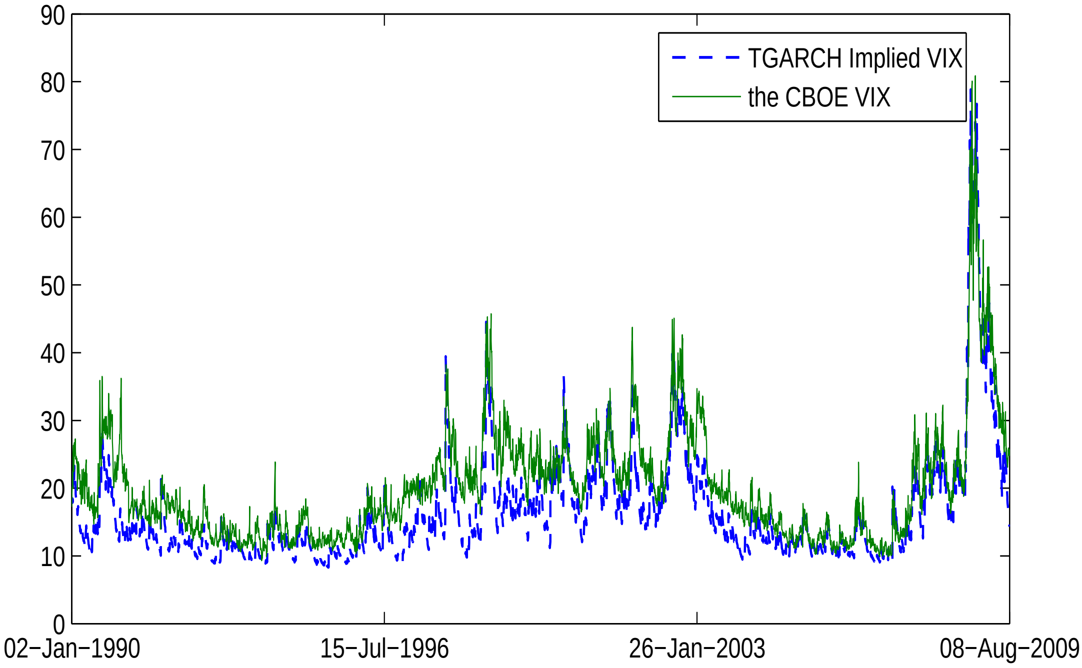

Figure 3: **The Comparison between TGARCH Implied VIX and the CBOE VIX from Jan 2, 1990 to Aug 8, 2009**
The estimated TGARCH(1,1) model is:
$h_t = 1.050 \times 10^{-6} + 0.001240\epsilon_{t-1}^2 + 0.1089\epsilon_{t-1}^2 1(\epsilon_{t-1} < 0) + 0.9333h_{t-1}$,
and estimated $\lambda = 0.02311$. The conditional variance of the first period is set at the variance of the rate of return of S&P 500 over the whole sample period, which is 18.63% on annual basis.

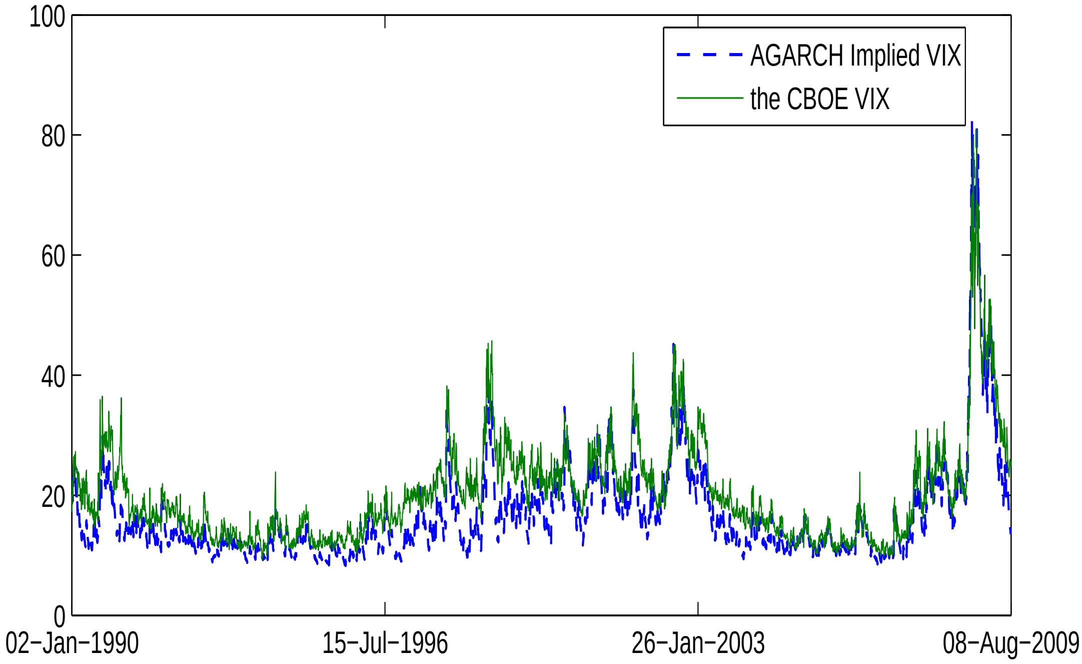

Figure 4: **The Comparison between AGARCH Implied VIX and the CBOE VIX from Jan 2, 1990 to Aug 8, 2009**
The estimated AGARCH(1,1) model is:
$h_t = 1.128 \times 10^{-6} + 0.05523(\epsilon_{t-1} - 1.014\sqrt{h_{t-1}})^2 + 0.8802h_{t-1}$,
and estimated $\lambda = 0.01498$.The conditional variance of the first period is set at the variance of the rate of return of S&P 500 over the whole sample period, which is 18.63% on annual basis.

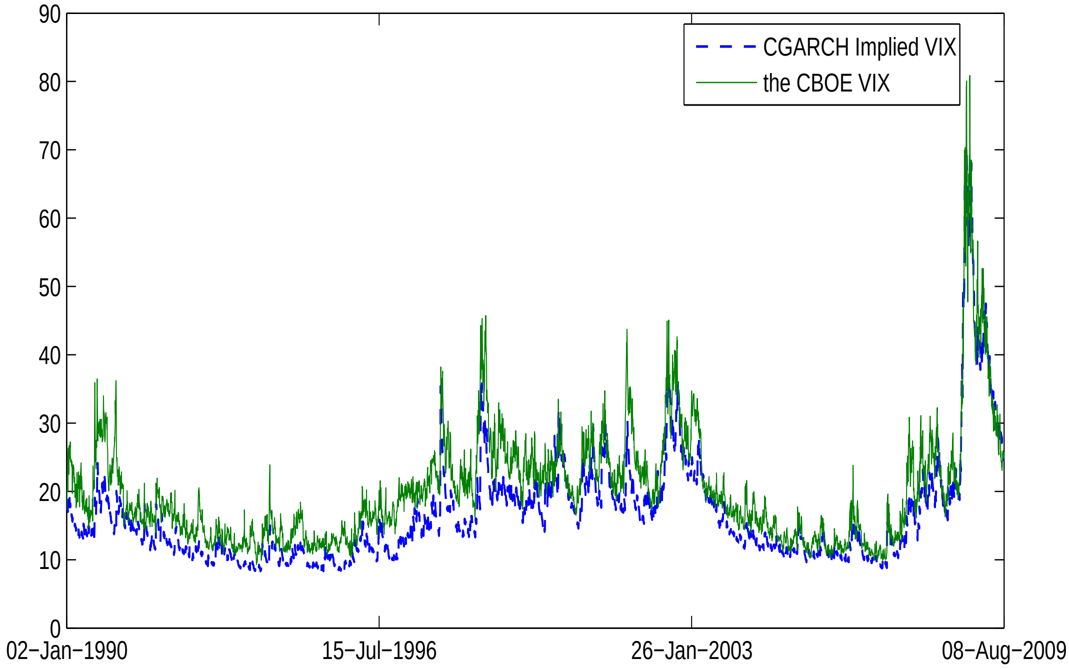

Figure 5: **The Comparison between CGARCH Implied VIX and the CBOE VIX from Jan 2, 1990 to Aug 8, 2009**
The estimated CGARCH(1,1) model is:
$h_t - q_t = 0.04814(\epsilon_{t-1}^2 - q_{t-1}) + 0.9169(h_{t-1} - q_{t-1})$,
$q_t = 2.010 \times 10^{-7} + 0.9984q_{t-1} + 0.02331(\epsilon_{t-1}^2 - h_{t-1})$,
and estimated $\lambda = 0.05286$. The transitory and permanent components of the conditional variance of the first period are both set at the variance of the rate of return of S&P 500 over the whole sample period, which is 18.63% on annual basis.